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INTRODUCTION TO THE EXERCISES OF CHAPTER 1

Important areas of current research can be analyzed very simply using the theory of relativity. This analysis depends heavily upon a physical intuition, which develops with experience. Such experience cannot be obtained in the laboratory—simple experiments in relativity are difficult and expensive because the speed of light is so great. As alternatives to simple experiments, the following exercises involve a wide range of physical consequences of the properties of spacetime. These properties of spacetime recur here over and over again in different contexts:

paradoxes
puzzles
derivations
technological applications
estimates
precise calculations
philosophical difficulties

The text of the chapter has presented all formal tools necessary to answer these exercises, but intuition—a practiced way of seeing—is best developed without hurry. For this reason it will prove useful to continue to do more and more of these exercises in relativity after one has moved on to material outside this book. Those who wish to cover the essential material in the least possible time may limit themselves to the exercises whose titles are set in boldface type in the list beginning below.

The mathematical manipulations in the exercises are very brief: only a few answers will take more than five lines to write down. On the other hand, the exercises will require some "rumination time." Unstarred exercises should require the least time; those marked with a single asterisk are more difficult; those marked with double asterisks are suitable for graduate students in physics.

WHEELER'S FIRST MORAL PRINCIPLE. *Never make a calculation until you know the answer.* Make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every puzzle. Courage: no one else needs to know what the guess is. Therefore make it quickly, by instinct. A right guess reinforces this instinct. A wrong guess brings the refreshment of surprise. In either case life as a spacetime expert, however long, is more fun!

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A. THE SPACETIME INTERVAL (Text Sections 5, 6, 7)

1. Space and time—a worked example

Two events occur at the same place in the laboratory frame of reference and are separated in time by 3 seconds. (a) What is the spatial distance between these two events in a rocket frame in which the events are separated in time by 5 seconds? (b) What is the relative speed β_r of the rocket and laboratory frames?

Solution: (a) The spacetime *interval* between these two events has the same value measured in either frame of reference

$$(\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - (\Delta x')^2$$

From the statement of the problem

$\Delta x = 0$
$\Delta t = 3 \text{ (seconds)} \times c \text{ (meters/second)} = 9 \times 10^8 \text{ meters}$

$\Delta x' = $ to be found
$\Delta t' = 5 \text{ (seconds)} \times c \text{ (meters/second)} = 15 \times 10^8 \text{ meters}$

Substitute these values into the expression for the interval

$$81 \times 10^{16} - 0 = 225 \times 10^{16} - (\Delta x')^2$$

From this equation find

$$(\Delta x')^2 = 144 \times 10^{16} \text{ meters}^2$$

or

$$\Delta x' = 12 \times 10^8 \text{ meters}$$

(b) In the *laboratory* frame the two events occur *at the same place*. In the *rocket* frame this laboratory "place" has moved 12×10^8 meters in 5 seconds—or in 15×10^8 meters of light-travel time. Therefore the relative speed of the two frames is

$$\Delta x'/\Delta t' = (12 \times 10^8)/(15 \times 10^8) = 4/5$$

2. Practical synchronization of clocks

You are an observer stationed near a clock with spatial coordinates $x = 6$ meters, $y = 8$ meters, and $z = 0$ meters in the laboratory frame. You wish to synchronize your clock with the one at the origin using the reference flash. Describe in detail and with numbers how to proceed.

3. Relations between events

Events A, B, and C are plotted in the laboratory spacetime diagram of Fig. 34. Answer the following questions for the pair of events A and B.

(a) Is the *interval* between the two events timelike, lightlike, or spacelike?

(b) What is the *proper time* (or *proper distance*) between the two events?

(c) Is it *possible* that one of the events *caused* the other event?

Answer the same questions for the pair of events A and C.

Answer the same questions for the pair of events C and B.

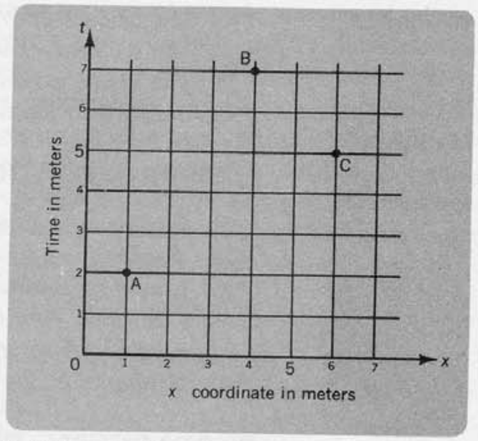


Fig. 34. What are the relations among the events A, B, and C?

4. Simultaneity

"A hits B and simultaneously one hundred million miles away C hits D." Explain in a sentence or two how special relativity teaches us to restate or qualify this statement.

5. Temporal order of events

"Event G occurred before event H." Prove that the *temporal order* of two events in the laboratory frame is the same as in all rocket frames if and only if the two events have either a timelike or a lightlike separation.

6.* The expanding universe

(a) A giant bomb explodes in otherwise empty space. What is the nature of the motion of one frag-

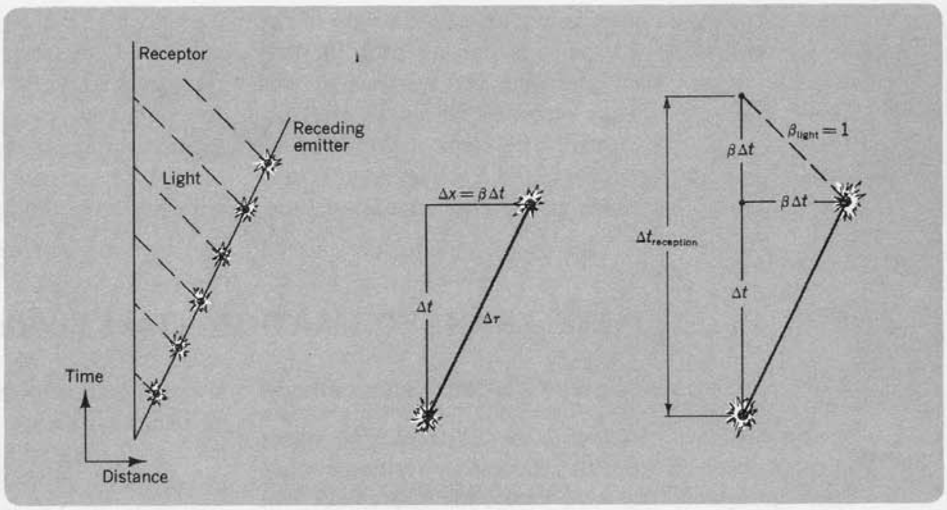


Fig. 35. Calculation of the time $\Delta t_{reception}$ between arrival at observer of consecutive flashes from receding emitter.

ment relative to another? And how can this relative motion be detected? *Discussion:* Imagine each fragment equipped with a beacon that gives off flashes of light at regular, known intervals $\Delta\tau$ of time as measured in its own frame of reference (proper time!). Knowing this interval between flashes, what method of detection can an observer on one fragment employ to determine the velocity β—relative to him—of any other fragment? Assume that he uses, in making this determination, (1) the known proper time $\Delta\tau$ between flashes and (2) *the time* $\Delta t_{reception}$ *between the arrival of consecutive flashes at his position.* (*Note:* This is *not* equal to the time Δt in his frame between the emission of the two flashes from the receding emitter; see Fig. 35.) Derive a formula for β in terms of $\Delta\tau$ and $\Delta t_{reception}$. How will the measured recession *velocity* depend upon the *distance* from one's own fragment to the fragment at which one is looking? (*Note:* In any given time in any given frame, fragments evidently travel *distances* in that frame from the point of explosion that are in direct proportion to their velocities in that frame!)

(b) How can observation of the light from stars be used to verify that the *universe is expanding*? *Discussion:* Atoms in hot stars give off light of different frequencies characteristic of these atoms ("spectral lines"). The *observed period* of the light in each spectral line from starlight can be measured on earth. From the *pattern* of spectral lines the kind of atom emitting the light can be identified. The same kind of atom can then be excited in the laboratory to emit light while at rest and the *proper period* of the light in any spectral line can be measured. Use the results of part (a) to describe how the *observed period* of light in one spectral line from starlight can be compared to the *proper period* of light in the same spectral line from atoms at rest in the laboratory to give the *velocity of recession* of the star that emits the light. This observed change in period due to the velocity of the source is called the *Doppler shift*. (For a more detailed treatment see Ex. 75 of Chap. 2 and the exercises which follow it.) *If* the universe began in a gigantic explosion, how must the observed velocities of recession of different stars at different distances compare with one another? Slowing down during expansion—by gravitational attraction or otherwise—is to be neglected here but is considered in more complete treatments (Ex. 80).

7. Proper time in communication

A flash of light is emitted from the sun and is absorbed on the moon. "The proper time between the emission of this flash and its absorption is equal to zero." True or false? Is the proper time between the two events (emission and absorption) equal to zero if the flash is reflected back and forth between mirrors on the moon before being absorbed? (Careful!) A flash of light is emitted on earth and travels *through air* directly to another spot on the earth, where it is absorbed. (The speed of light in air is slightly less than c.) Is the proper time between the emission of this flash and its absorption equal to zero?

8. Data-collecting and decision-making

We have used a latticework of recording clocks to describe events. The position of an event is the position of the clock nearest to the event, and the time of an event is the time recorded on that clock. Physics deals with the study of the *relations* between events. If the data-analysis center is located at the origin of the latticework of clocks, what is the lag time (in that frame) between data available for analysis at this center and data already recorded on clocks at a distance R from that center? The clock at $x = 6 \times 10^9$ meters, $y = 8 \times 10^9$ meters, and $z = 0$ meters records the passage of a meteor at 41×10^9 meters of time. The clock at $x = 3 \times 10^9$ meters, $y = 4 \times 10^9$ meters, and $z = 0$ meters records the passage of the same meteor at 47×10^9 meters of time. The observers in the data-analysis center require 3 seconds to take evasive action. If the data above are sent to them by light flash and are displayed instantly upon arrival, will they have time to protect themselves?

B. THE LORENTZ TRANSFORMATION (Text Sections 8 and 9)

9. Lorentz contraction—a worked example

A meter stick is attached to a rocket. The meter stick is observed from the laboratory frame of reference (laboratory framework of rods and clocks). In what way will the findings of the laboratory observer about the length of the meter stick contrast with those predicted by pre-relativity physics? We break this broad question down into four parts:

(a) How can this question about *length* be translated into a question about the separation of two *events?* Remarks: Each end of the meter stick traces out a world line through spacetime. But one world line is an infinite succession of events. So how is one going to pick out, in a reasonable way, exactly two events that will give the desired information about the apparent length of the meter stick?

> *Solution:* Select the following two events for attention. A: One end of the meter stick flashes past a laboratory clock just as that clock reads noontime. B: The other end of the meter stick flashes past another laboratory clock when it too reads noontime. *Discussion:* One must measure the location of both ends of the moving meter stick at the same time in the laboratory frame. Otherwise there would not be a well defined pair of laboratory points between which to carry out the length measurement. The two events are thus simultaneous in the laboratory frame of reference ($\Delta t = 0$). They may or may not be simultaneous in the rocket frame ($\Delta t'$ may or may not be zero). No matter! The meter stick is at rest in the rocket frame. In that frame the two ends may be located at leisure.

(b) When the meter stick points along the x axis (direction of motion) of the rocket so that the separation of the two ends in the rocket frame is $\Delta x' = 1$ meter, what length is observed in the laboratory frame?

> *Solution:* The length is the separation in space of the two events A and B in the laboratory frame
>
> $$\Delta x = \Delta x'/\cosh \theta_r = \Delta x'(1 - \beta_r^2)^{1/2} \tag{38}$$
>
> This length is less than one meter. The shortening is called the *Lorentz contraction.* Discussion: The Lorentz transformation (Eqs. 37) connects separations in the laboratory frame with separations in the rocket frame by the equations
>
> $$\begin{aligned} \Delta x' &= \Delta x \cosh \theta_r - \Delta t \sinh \theta_r \\ \Delta t' &= -\Delta x \sinh \theta_r + \Delta t \cosh \theta_r \\ \Delta y' &= \Delta y \\ \Delta z' &= \Delta z \end{aligned} \tag{39}$$
>
> The two events are simultaneous in the laboratory frame ($\Delta t = 0$). Therefore $\Delta x' = \Delta x \cosh \theta_r$, from which the answer follows. Note that $\Delta t'$ is not equal to zero; that is, the events A and B are not simultaneous as recorded in the rocket frame. This difference in time between events at the two ends of the meter stick raises no questions in the minds of the rocket workers as to the length of the meter stick. To them it is at rest and it is one meter long. Neither are they troubled that the laboratory observers record the length as shortened ("Lorentz contracted"). "Why not?" they say. "The laboratory observers marked down the positions of the two ends of the meter stick at times, t_A' and t_B', that we know to be different. How could they help but get a length different from 1 meter?"

(c) When the meter stick points along the y axis (perpendicular to the direction of motion) of the rocket frame, so that the separation of the two ends in the rocket frame is $\Delta y' = 1$ meter, what length is observed in the laboratory frame?

> *Solution:* The length is the separation in space of the two events A and B in the laboratory frame
>
> $$\Delta y = \Delta y'$$

This length is 1 meter. There is no shortening of dimensions perpendicular to the direction of motion. Discussion: Note that the two events are now simultaneous not only in the laboratory frame ($\Delta t = 0$), but also in the rocket frame ($\Delta t' = 0$; see Eqs. 39). Thus it is not surprising to the rocket workers that the laboratory observers should agree with them about the length of the meter stick.

(d) Reconsider the conclusion of part b. How can one possibly accept the result that a rocket meter stick appears to be shorter than one meter to laboratory observers? If this conclusion were true, would we not have a way to distinguish the physics in the rocket frame (where meter sticks have their standard length) from the physics in the laboratory (where the same meter sticks are recorded as shortened)? And if so, does not the reasoning of relativity destroy the very foundation principle of relativity? This principle states that one cannot distinguish between one inertial frame and another by any difference between the physics in the two frames. Have we not found a most striking difference between the physics in the two frames?

Solution: Yes, there is a difference between the x dimensions recorded in the two frames; but there is no difference between the *physics* in the two frames. A meter stick that is at rest relative to the rocket and that points along the direction of motion, is recorded as shorter than 1 meter in the laboratory. However, a meter stick that is at rest in the *laboratory* and that is parallel to the direction of motion is recorded as also shortened by the *rocket* workers. Objection: What preposterous story is this! I will stick to simple logic and defy all this relativity nonsense. You say that a rocket meter stick may be recorded in the laboratory as a half meter. Then you must agree that a length of a half meter in the laboratory is recorded as a full meter in the rocket frame. So rocket dimensions are longer than laboratory dimensions—along the direction of motion. Physics is as different as it could well be between the two frames. I would have no trouble at all telling whether I was in the laboratory frame or the rocket frame. Principle of relativity! What delusion! Reply: Perhaps all of us find Einstein and Lorentz disturbing at a first encounter because we have had so little experience with objects moving at really high velocities. Perhaps you will feel happier with the principle of relativity if you see its analog in Euclidean geometry. Of course there are some differences between the formula $(\Delta L)^2 = (\Delta x)^2 + (\Delta y)^2$ in Euclidean geometry and $(\Delta \tau)^2 = (\Delta t)^2 - (\Delta x)^2$ in Lorentz geometry. However, the question whether distances are different in two frames clearly worries you more than the question whether the distance in the new frame is less than in the old frame (Lorentz contraction in Lorentz geometry) or greater (length increase in Euclidean geometry). So look at Fig. 36. A pasture that extends for the distance $\Delta x'$ in the x' direction evidently extends for a greater distance in the x direction

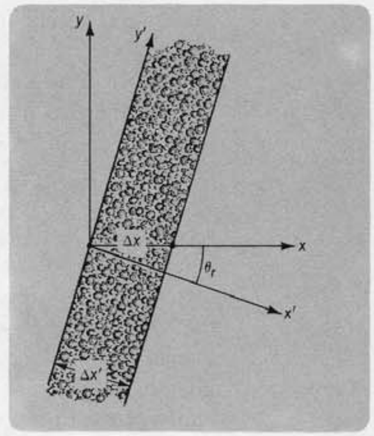


Fig. 36. A pasture extends for a greater distance in the x direction than in the x' direction.

(40) $$\Delta x = \Delta x'/\cos \theta_r$$

On the other hand, look now at Fig. 37 (see Ex. 48 for space and time analogs of Figs. 36 and 37). Here there is another field, which extends for the distance Δx in the x direction. However, its extension in the x' direction is greater

(41) $$\Delta x' = \Delta x/\cos \theta_r$$

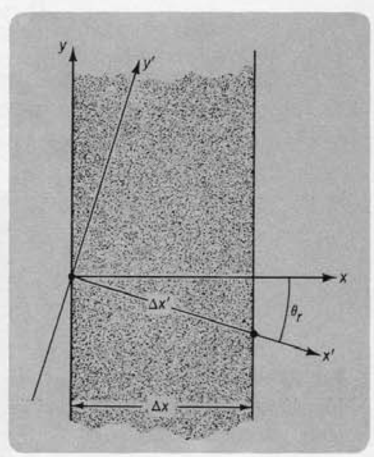


Fig. 37. Another field extends for a greater distance in the x' direction than in the x direction.

Surely you accept these results. You do not even worry about any inconsistency between formulas 40 and 41. You know as well as anyone that the Δx's in the two formulas refer to *different measurements* on *different fields*. So perhaps you will be willing to believe that the length of a meter stick that is at rest relative to a rocket will be recorded as less than a meter in the laboratory, whereas a meter stick that is at rest in the laboratory will be less than a meter to the recorders on the rocket. Response: I now agree that there is no logical inconsistency in what you have been telling me. But perhaps you will go a step further and really prove to me what you have just now said about a laboratory meter stick being recorded as less than a meter in the rocket frame. Answer: Solve the Lorentz transformation equations (Eqs. 39) for the coordinates in the laboratory frame in terms of the coordinates in the rocket frame; or merely interchange the role of the primed and unprimed coordinates in those equation, and reverse the sign of the velocity; or look up Eqs. 36, inverse to Eqs. 39; in any case, write down the relations

$$\begin{aligned} \Delta x &= \Delta x' \cosh \theta_r + \Delta t' \sinh \theta_r \\ \Delta t &= \Delta x' \sinh \theta_r + \Delta t' \cosh \theta_r \\ \Delta y &= \Delta y' \\ \Delta z &= \Delta z' \end{aligned} \tag{42}$$

Our new meter stick is at rest in the laboratory frame. It is moving as viewed from the rocket frame. Consequently a determination of its length in the rocket frame requires us to have in the rocket frame two fiducial points: the locations of the two ends of the meter stick at the same rocket time. Thus $\Delta t' = 0$. From the first of Eqs. 42 we find immediately

$$\Delta x' = \Delta x/\cosh \theta_r = \Delta x(1 - \beta^2)^{1/2} \tag{43}$$

The length recorded in the rocket frame is less than one meter when the meter stick is at rest in the laboratory—as was to be shown.

10. Time dilation

A clock is carried by a rocket (Fig. 38). The clock is observed from the laboratory frame of reference (laboratory latticework of rods and clocks). In what way will the findings of the laboratory observer about the time readings of the traveling clock contrast with those predicted by pre-relativity physics? Break this question down into four parts.

(a) How can this question about time lapse be translated into a question about the separation of two *events?*

(b) Let the rocket clock read one meter of light-travel time between the two events chosen in part a, so that the lapse of time recorded in the rocket frame is $\Delta t' = 1$ meter. Show that the time lapse observed in the laboratory frame is given by the expression

$$\Delta t = \Delta t' \cosh \theta_r = \Delta t'/(1 - \beta^2)^{1/2} \tag{44}$$

This time lapse is *more* than one meter of light-travel time. Such lengthening is called *time dilation* ("to dilate" means "to stretch").

(c) How can one possibly accept the conclusion of part b that one meter of rocket time appears longer than one meter to laboratory observers? Does not this result give one a way to distinguish the physics in the rocket frame (where clocks run at their standard rate) from the physics in the laboratory frame (where the same clocks are recorded as running slow)? Therefore does not this reasoning violate the *principle of relativity* (Sect. 3) on which rests the whole theory of relativity?

(d) Go one step further and show that one meter of time as recorded by a clock carried in the laboratory frame ($\Delta t = 1$ meter) is recorded as more than one meter of time by observers in the rocket frame, according to the formula

$$\Delta t' = \Delta t \cosh \theta_r = \Delta t/(1 - \beta^2)^{1/2} \tag{45}$$

In what way does this result verify the symmetry between laboratory and rocket frames required by the principle of relativity?

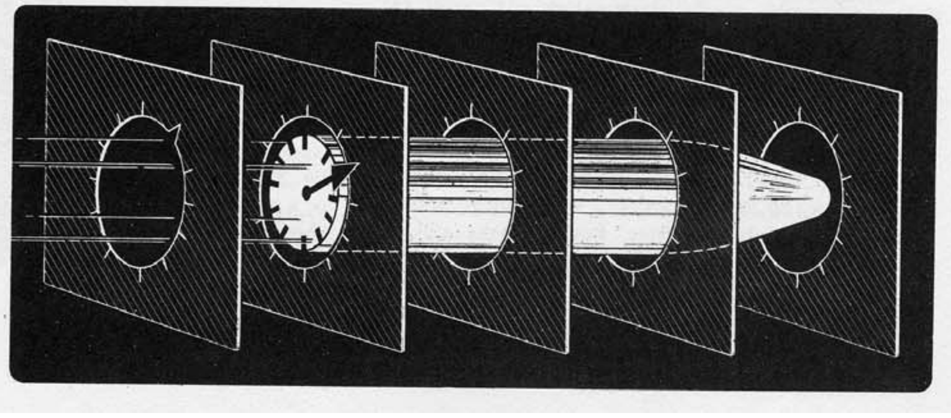


Fig. 38. A method for comparing several laboratory clocks with one rocket clock.

11. Relative synchronization of clocks

(a) Show that if two events occur simultaneously *and at the same place* in the laboratory frame they will occur simultaneously in all rocket frames. Show that if two events occur simultaneously in the laboratory frame but *not at the same position* on the x axis of the laboratory frame, they will *not* be simultaneous as observed in any moving rocket frame. The fact that observers in relative motion do not always agree whether two events are simultaneous is called the *relativity of simultaneity*.

(b) Two events occur simultaneously and at the same x coordinate in the laboratory frame, but are separated by the y and z coordinates Δy and Δz. Show that these two events are also simultaneous in the rocket frame.

(c) Use the Lorentz transformation equation to show that at $t = 0$ in the laboratory frame the clocks along the positive x axis in the rocket frame appear to be set behind those in the laboratory frame, with clocks farther from the origin set farther behind; and that clocks along the negative x axis in the rocket frame appear to be set ahead of those in the laboratory frame, with clocks farther from the origin set farther ahead, according to the equation

$$t' = -x \sinh \theta_r = -x\,\beta_r/(1 - \beta_r^2)^{1/2} \tag{46}$$

(d) Use the inverse Lorentz transformation equation to show that at $t' = 0$ in the rocket frame the clocks along the positive x axis in the laboratory frame appear to be set ahead of those in the rocket frame, with clocks farther from the origin set farther ahead; and that clocks along the negative x axis in the laboratory frame appear to be set behind those in the rocket frame, with clocks farther from the origin set farther behind, according to the equation

$$t = +x' \sinh \theta_r = +x'\,\beta_r/(1 - \beta_r^2)^{1/2} \tag{47}$$

The fact that neither of two observers in relative motion agrees that the reference event and the reading of zero time on all clocks of the *other* frame occur simultaneously is called the *relative synchronization* of clocks.

(e) The difference in sign between the equations in parts c and d seems to imply an asymmetry between frames that might be used to tell them apart—which would violate the principle of relativity. Show that if an observer in *either* frame chooses his positive x axis to lie in the direction of motion of the other frame, then physical measurements on the synchronization of clocks will give results in the two frames which are indistinguishable. In other words, the two frames themselves are indistinguishable using this method. The difference in sign between the above equations is due to an arbitrary—and asymmetric—choice of a *common* direction for both positive x axes.

(f) The foregoing results are sometimes summarized by stating that a "rocket observer sees the laboratory clocks to be out of synchronism with one another." Explain what is wrong with this way of stating the matter. Show that a single rocket observer is not enough to make the required measurements. What is a sharp, clean, legalistically correct, and clear (even if considerably longer!) way to state the same result?

12. Euclidean analogies

(a) A straight rod lies in the xy plane of a Euclidean coordinate system. Draw a diagram showing the rod in the xy plane; label the projections of this rod on the x, y and x', y' axes. Spell out an explicit analogy between the x components of the length of this rod as measured in two rotated Euclidean coordinate systems and the different lengths of a moving rod observed in the laboratory frame and in the rocket frame in which the rod is at rest.

(b) Spell out an explicit analogy between time dilation and the y components of length of the rod of part a as observed in two rotated Euclidean coordinate systems. What are the Euclidean and Lorentz invariants?

(c) Spell out an explicit analogy between the relative synchronization of clocks and the case of two rotated Euclidean coordinate systems in which points on the positive x axis of *one* coordinate system have, say, a negative y coordinate in the *other* coordinate system (more negative for points farther from the common origin).

13. Lorentz contraction II

A meter stick lies along the x' axis and at rest in the rocket frame. Show that an observer in the laboratory frame will conclude that the meter stick has undergone Lorentz contraction if he measures how long it takes the meter stick to pass one of his clocks and multiplies this result by the relative velocity of the two frames.

14. Time dilation II

Two events occur at the same place but at different times in the rocket frame. Show that an observer in the laboratory frame will conclude that the time between the two events has been dilated if he measures the distance between them in the laboratory frame and divides this distance by the relative velocity of the two frames.

15. Lorentz transformation equations with time in seconds

If time is expressed in seconds (written with a subscript: t_{sec}) and if v_r represents the relative speed between laboratory and rocket frames expressed in meters per second, show that the Lorentz transformation equations become

$$x' = x \cosh \theta_r - ct_{\text{sec}} \sinh \theta_r = \frac{x - v_r t_{\text{sec}}}{(1 - v_r^2/c^2)^{1/2}}$$

(48)

$$t'_{\text{sec}} = -(x/c) \sinh \theta_r + t_{\text{sec}} \cosh \theta_r = \frac{t_{\text{sec}} - (v_r/c^2)x}{(1 - v_r^2/c^2)^{1/2}}$$

where

$$v_r/c = \tanh \theta_r.$$

Write down the *inverse* Lorentz transformation equations using the same notation.

16.* Derivation of the Lorentz transformation equations

Derive the transformation equations of Lorentz along new lines (due to Einstein) as follows. Let the rocket move uniformly with velocity β_r in the x direction of the laboratory. The coordinates x', y', z', t' of any event, such as an explosion, in the rocket reference frame have a one-to-one relation with the coordinates x, y, z, t of the same event measured in the laboratory frame. Moreover, $y = y'$ and $z = z'$ (perpendicular distances are the same). As for the relation between x, t and x', t', assume a *linear* relationship

$$x = ax' + bt'$$
$$t = ex' + ft'$$

with four coefficients a, b, e, f that (1) are unknown, (2) are independent of x, t and x', t', and (3) depend only upon the relative velocity β_r of the two frames of reference.

Find the *ratios* b/a, e/a, f/a as functions of velocity β_r using the following three arguments and these arguments alone: (1) A flash of light that starts at $x = 0$, $t = 0$ ($x' = 0$, $t' = 0$) moves to the *right* at the velocity of light ($x = t$; $x' = t'$) in *both* frames of reference. (2) A flash of light that starts at $x = 0$, $t = 0$ ($x' = 0$, $t' = 0$) moves to the *left* at the velocity of light ($x = -t$; $x' = -t'$) in *both* frames of reference. (3) The point $x' = 0$ has the velocity β_r in the laboratory frame.

Now use as the fourth piece of information, the invariance of the interval (Section 5): (4) $t^2 - x^2 = (t')^2 - (x')^2$ to find the constant a itself and thus all four coefficients a, b, e, f. Do the results obtained in this way agree with *Lorentz's* values for the transformation coefficients?

17.* Proper distance and proper time

(a) Two events P and Q have a spacelike separation. Show that a rocket frame can be found in which the two events occur at the *same time*. Also show that in this rocket frame the distance between the two events is equal to the proper distance σ between them. (One method: assume that such a rocket frame exists and then use the Lorentz transformation equations to show that the relative velocity of this rocket frame is less than the speed of light ($\beta_r < 1$), thus justifying the assumption made.)

(b) Two events P and R have a timelike separation. Show that a rocket frame can be found in which the two events occur at the *same place*. Also show that in this rocket frame the time between the two events is equal to the proper time τ between them.

18.* The place where both agree

At any instant there is just one plane in which both the laboratory and the rocket clocks agree. Show that the velocity of this plane in the laboratory frame is equal to $\tanh(\theta_r/2)$, where θ_r is the relative velocity parameter between laboratory and rocket frames.

19.* Transformation of angles

A meter stick lies at rest in the rocket frame and makes an angle ϕ' with the x' axis. What angle ϕ does the same meter stick make with the x axis of the laboratory frame? What is the *length* of the meter stick as observed in the laboratory frame? Next *assume* that the directions of electric-field lines around a point charge transform in the same way as the directions of meter sticks that lie along these lines. Draw qualitatively the electric-field lines due to an isolated positive point charge at rest in the rocket frame as seen in (a) the rocket frame and (b) the laboratory frame. What conclusions follow concerning the forces exerted, in the laboratory frame, on stationary test charges that surround a charge moving in that frame?

20.* Transformation of y velocity

A particle moves with uniform speed $\beta^{y'} = \Delta y'/\Delta t'$ along the y' axis of the rocket frame. Transform the components of y and t displacements using the Lorentz transformation equations. Show that the x component and the y component of the velocity of this particle in the laboratory frame are given by the expressions

$$\begin{aligned} \beta^x &= \tanh \theta_r \\ \beta^y &= \beta^{y\prime}/\cosh \theta_r \end{aligned} \tag{49}$$

21.** Transformation of velocity directions

A particle moves with a velocity β' in the $x'y'$ plane of the rocket frame in a direction that makes an angle ϕ' with the x' axis. Find the angle that the velocity vector of this particle makes with the x axis of the laboratory frame. (Hint: Transform displacements rather than velocities.) Why does this angle differ from that found in Ex. 19? Contrast the two results when the relative velocity between the rocket and laboratory frames is very great.

22.** The headlight effect

A flash of light is emitted at an angle ϕ' with respect to the x' axis of the rocket frame. Show that the angle ϕ that the direction of this flash makes with respect to the x axis of the laboratory frame is given by the equation

$$\cos \phi = \frac{\cos \phi' + \beta_r}{1 + \beta_r \cos \phi'} \tag{50}$$

Show that your answer to the previous exercise gives the same result when the velocity β' is given the value one. Now consider a particle at rest in the rocket frame that emits light uniformly in all directions. Consider the 50 percent of this light that goes into the *forward* hemisphere in the rocket frame. Also, assume that the rocket moves very fast relative to the laboratory. Show that in the laboratory frame this light is concentrated in a *narrow forward cone* whose axis lies in the direction of motion of the particle. This effect is called the *headlight effect*.

C. PUZZLES AND PARADOXES

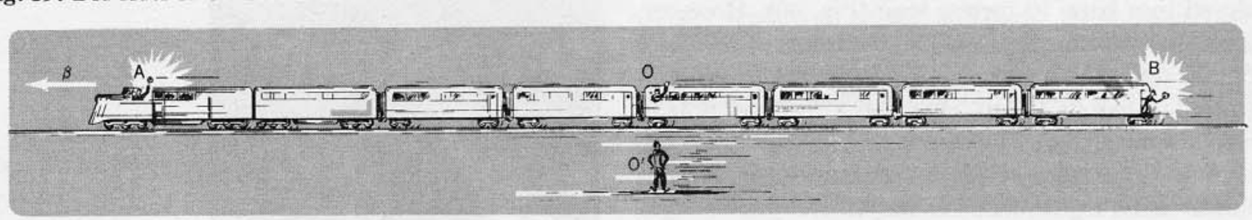
23. Einstein's train paradox—a worked example

Three men (A, O, and B) are riding on a train at a velocity β_r close to the value one. A is in front, O is at the middle, and B is at the rear (Fig. 39). A fourth man O′ is standing beside the rails. At the very instant O passes O′ it happens that two flash-bulb signals coming from A and B reach O and O′. Who emitted the signal first? Using *only* the fact that the speed of light is finite and independent of the source velocity, show that O and O′ give different answers to this question. Having answered this qualitative question, evaluate the quantitative difference between the times of emission of the flashes from A and B as observed in the frame of reference of the train (Δt_{BA}) and the frame of reference of O′ ($\Delta t_{BA}'$), either by the Lorentz transformation or by other means.

Solution: Observers A and B are *at rest* with respect to observer O. They are also equidistant from observer O, as he can verify with a meter stick at his leisure. Therefore flashes from A and B require *equal times* to arrive at O. Flashes from A and B are observed to arrive at O at the same time. *Observer O concludes, therefore, that observers A and B* ***emitted*** *their flashes at the same time*: $\Delta t_{BA} = 0$.

Observer O′ standing beside the rails draws an entirely different conclusion. He reasons as follows: "The two flashes arrived when the middle of the train (observer O) was passing me. Therefore the two flashes must both have been emitted *before* the middle of the train reached me. Before the middle of the train reached me, observer A was *nearer to me* than was observer B. Thus

Fig. 39. Did rider A or rider B emit his flash first?



light from B had farther to travel to reach me—and therefore *took a longer time* to reach me—than did light from A. But both flashes *arrived* at the same time. Therefore *observer B must have emitted his flash* ***before*** *observer A emitted his flash.*" ($\Delta t'_{BA} = t'_B - t'_A < 0$) In summary, observer O′ beside the tracks concludes that B emitted his flash *before* A emitted his flash, while observer O riding on the train concludes that A and B emitted their flashes *at the same time*.

What is the observed difference of times between the emissions, A and B, of these flashes? In the unprimed train frame the flashes are emitted simultaneously, so $\Delta t = 0$. The separation between the emissions is $\Delta x = \Delta x_{BA} = x_B - x_A = L$ where L is the length of the train. Therefore in the primed frame (which moves to the *right* relative to the unprimed train frame, as is conventional in the primed-unprimed notation) the time between emissions can be found from the Lorentz transformation equation

$$\Delta t' = -\Delta x \sinh \theta_r + \Delta t \cosh \theta_r$$
$$\Delta t' = -L \sinh \theta_r = -L\beta_r/(1 - \beta_r^2)^{1/2}$$

The minus sign shows that observer B (who is on the positive x' axis) emitted his flash at an *earlier*—a more negative—rocket time than observer A emitted his flash.

24. Einstein puzzler

When Einstein was a boy, he mulled over the following puzzler: A runner looks at himself in a mirror that he holds at arm's length in front of him. If he runs with nearly the speed of light, will he be able to see himself in the mirror? Analyze this question in terms of relativity.

25.* The pole and barn paradox

A worried student writes, "Relativity must be *wrong*. Consider a 20-meter pole carried so fast in the direction of its length that it appears to be only 10 meters long in the laboratory frame of reference. Therefore at some instant the pole can be entirely enclosed in a barn 10 meters long (Fig. 40). However, look at the same situation from the frame of reference of the runner. To him the *barn* appears to be contracted to half its length. How can a 20-meter pole fit into a 5-meter barn? Does not this unbelievable conclusion prove that relativity contains somewhere a fundamental logical inconsistency?"

Write a reply to the worried student explaining

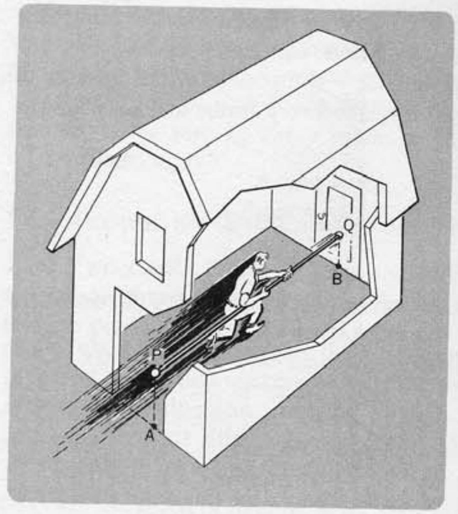


Fig. 40. Fast runner with "20-meter pole" enclosed in "10-meter barn." In the next instant he will burst through the back door, which is made of paper.

clearly and carefully how the pole and barn are treated by relativity without contradiction. (Clear up the paradox by making two carefully labeled space-time diagrams, one an xt diagram, the other an $x't'$ diagram. Take the "event" Q coinciding with A to be at the origin of both diagrams. In both plot the world lines of A, B, P, and Q. Pay attention to the *scale* of both diagrams. Label both diagrams with the times (in meters) at which Q coincides with B. Do the same for the times at which P coincides with B. Calculate these times, using the equations of the Lorentz transformation or some other method.)

26.** Space war

Two rockets of equal rest length are passing "head on" at relativistic speeds. Observer O has a gun in the

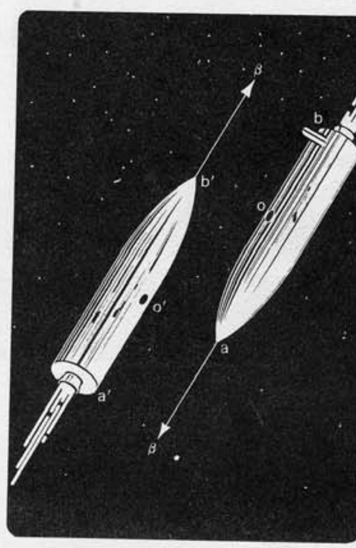


Fig. 41. Two rocket ships passing at high speed.

tail of his rocket pointing perpendicular to the direction of relative motion. He fires the gun when points a and a′ coincide:

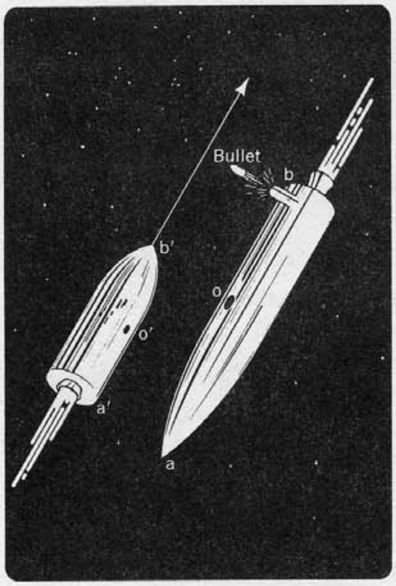


Fig. 42. In frame of *o* one expects a bullet fired when *a* coincides with *a′* to *miss* other ship.

In the frame of O the other rocket ship is Lorentz contracted. Therefore O expects his bullet to miss the other rocket. But in the frame of the other observer, O′, it is the rocket ship of O that appears to be Lorentz contracted. Therefore when points a and a′ coincide, observer O′ sees

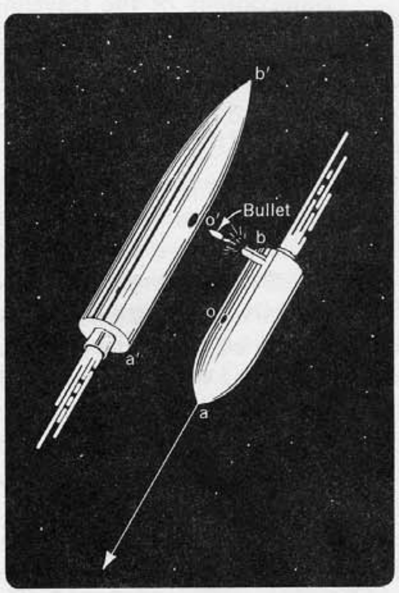


Fig. 43. In frame of *o′* one expects a bullet fired when *a* coincides with *a′* to *hit* other ship.

Does the bullet actually hit or miss? Discuss your answer: Pinpoint the looseness of the language used to state the problem, also the error in one diagram.

27.* The clock paradox†

Version I; other versions in Exs. 49, 51, and 81

On their twenty-first birthday, Peter leaves his twin Paul behind on the earth and goes off in the x direction for seven years of *his* time (2.2×10^8 seconds or 6.6×10^{16} meters of time) at $(24/25) = 0.96$ the speed of light, then reverses direction and in another seven years of his time returns at the same speed. (a) What is Peter's age on his return? (b) Make a spacetime diagram showing the motion of Peter. Indicate on it the x and t coordinates of the turn-around point and of the point of reunion. For simplicity idealize the earth as an inertial frame, adopt this inertial frame in the construction of the diagram, and take the origin to be the event of departure. (c) How old is Paul at the moment of reunion?

28.* Things that move faster than light‡

The Lorentz transformation equations have no meaning if the relative velocity of the two frames is greater than the velocity of light. This is taken to imply that mass, energy, and information (messages) cannot be moved from place to place faster than the speed of light. Check this implication in the following examples.

(a) The scissors paradox. A very long straight rod, which is inclined at an angle ϕ with the x axis, moves downward with uniform speed β^y (Fig. 44). Find the speed β_A of the point of intersection A of the lower edge of the stick with the x axis. Can this speed be greater than the speed of light? Can it be used to transmit a *message* from the origin to someone far out on the x axis?

(b) Suppose that the same rod is initially at rest with the point of intersection A at the origin. The region of the rod which is centered on the origin is struck by the downward blow of a hammer. The point of intersection moves to the right. Can this motion of the point of intersection be used to transmit a message faster than the speed of light?

†For reprints of several articles on the clock paradox, together with references to many more articles, see *Special Relativity Theory*, Selected Reprints, published for the American Association of Physics Teachers by the American Institute of Physics, 335 East 45th Street, New York 17, New York, 1963.

‡See Milton A. Rothman, "Things that go Faster than Light," Scientific American **203**, 142 (July, 1960).

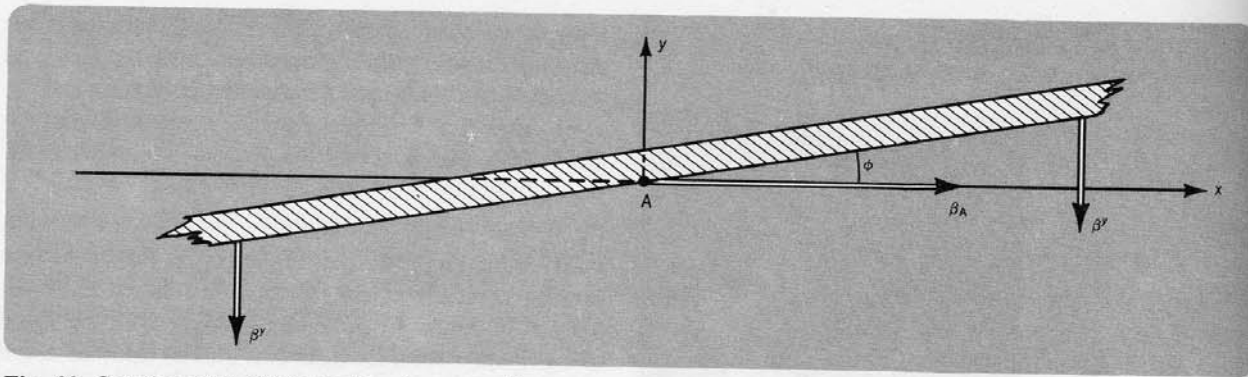


Fig. 44. Can the point of intersection A move with a speed greater than the speed of light?

(c) A very powerful searchlight is rotated rapidly in such a way that its beam sweeps out a flat plane. Observers A and B are on the plane and each the same distance from the searchlight but not near to each other. How far from the searchlight must A and B be in order that the searchlight beam will sweep from A to B faster than a light signal could travel from A to B? Before they took their positions, the two observers were given the following instructions:

To A: "When you see the searchlight beam, fire a bullet at B."

To B: "When you see the searchlight beam, duck because A has fired a bullet at you."

Under these circumstances, has not a warning gone from A to B with a speed faster than that of light?

(d) The manufacturers of some oscilloscopes claim writing speeds in excess of the speed of light. Is this possible?

D. BACKGROUND

29. Synchronization by a traveling clock—a worked example

Mr. Engelsberg does not approve of our method of synchronizing clocks by light flashes (Sect. 4). "I can synchronize my clocks in any way I choose," he says. Is he right? Mr. Engelsberg wishes to synchronize two identical clocks, named Big Ben and Little Ben, which are separated by one million miles (a little more than 1.5×10^9 meters) and which have zero relative velocity. He uses a third clock, identical in construction to the first two, that travels with constant velocity between them. As his moving clock passes Big Ben, it is set to read the same time as Big Ben. When the moving clock passes Little Ben, that outpost clock is set to read the same time as the traveling clock. "Now Big Ben and Little Ben are synchronized," says Mr. Engelsberg. Is he right? How much out of synchronism are Big and Little Ben as measured by a latticework of clocks—at rest relative to them both—that has been synchronized in the conventional manner using light flashes? Evaluate this lack of synchronism when the traveling clock that Mr. Engelsberg uses moves at one hundred thousand miles per hour (4.5×10^4 meters per second). Is there any earthly reason—aside from matters of personal preference—why we all should not adopt the method of synchronization used by Mr. Engelsberg?

Solution: Start with the numerical part of the solution. A latticework of clocks at rest with respect to Big Ben and Little Ben—a latticework whose clocks are synchronized by the standard method using light flashes—can be used to make observations of the traveling clock. Relative to this latticework the traveling clock moves at $v = 4.5 \times 10^4$ meters per second, or $\beta = v/c =$

$$\frac{4.5 \times 10^4 \text{ meters per second}}{3 \times 10^8 \text{ meters per second}} = 1.5 \times 10^{-4} \text{ meters}$$

of distance per meter of light-travel time. At this rate it covers the distance between Big Ben and Little Ben in a time, $\Delta t = 10^{13}$ meters of light-travel time. Comparison of readings of the lattice clocks with the traveling clock as it passes these in turn will show the phenomenon of *time dilation* (Ex. 10). With respect to the lattice clocks the traveling clock will run slow by a factor $(1 - \beta^2)^{1/2}$. Therefore, the time, $\Delta t'$, of travel between Big Ben and Little Ben as recorded on the *traveling clock* is

$$\begin{aligned}\Delta t' &= \Delta t(1 - \beta^2)^{1/2}\\ &= \Delta t(1 - 2.25 \times 10^{-8})^{1/2}\end{aligned}$$

Use the binomial expansion

$$(1 - \delta)^{1/2} = 1 - (\delta/2) - (1/8)\delta^2 - \cdots \approx 1 - \delta/2 \quad \text{(for small } \delta\text{)}$$

to give an approximate answer

$$\Delta t' \approx \Delta t - (1/2)\,2.25 \times 10^{-8}\,\Delta t$$

or

$$(51) \qquad \Delta t' - \Delta t = -1.12 \times 10^{-8} \times 10^{13}$$
$$= -1.12 \times 10^{5} \text{ meters} = -0.4 \times 10^{-3} \text{ seconds}$$

Set Little Ben by the traveling clock and then compare its reading with nearby clocks of the lattice. Little Ben will then read *earlier* than the lattice clocks by 0.4 millisecond.

There is a more direct way to find the lapse of time $\Delta t'$ recorded by the traveling clock on its way from Big Ben to Little Ben. The route is straight. The lapse of traveling-clock time along this world line is therefore equal to the proper length of the world line itself between the two events; that is, equal to the *interval* between the passages past Big and Little Ben:

$$\Delta t' = \Delta(\text{proper time}) = (\text{interval}) = [(\Delta t)^2 - (\Delta x)^2]^{1/2}$$

This calculation gives the result for the time discrepancy between laboratory clocks and the traveling clock

$$\Delta t' - \Delta t = [(\Delta t)^2 - (\Delta x)^2]^{1/2} - \Delta t$$

in complete agreement with Eq. 51.

Now return to consider the validity of the traveling-clock method of defining synchronization of clocks. Mr. Engelsberg is free to define synchronization in any way he wishes. However, if he uses the traveling-clock method to synchronize Big Ben and Little Ben he will find the following difficulties: (1) The settings one gives to laboratory clocks by this method of synchronization will depend on the *speed* of the traveling clock. Let the traveling clock move ten times faster than the speed given in the example above. Then the discrepancy between Little Ben and nearby lattice clocks will be not 0.4 milliseconds but about 40 milliseconds. Two Little Bens side by side that are synchronized using traveling clocks moving at different speeds will not agree with each other! (2) Even if traveling clocks are limited to a given speed, the results of this method of synchronization will depend on the *path* of the traveling clock. The longer the path taken by the traveling clock at its fixed speed, the earlier will Little Ben read as compared to nearby lattice clocks. (3) If the traveling clock makes a *round trip* from Big Ben, it will not be synchronized with Big Ben on its return! (The clock paradox, Ex. 27.) There are other inconveniences that result from Mr. Engelsberg's method of synchronization, but these are enough to show its inappropriateness for any simple description of what goes on in spacetime.

30. Time dilation and construction of clocks

In describing the phenomenon of time dilation in Ex. 10 we made no distinction between spring clocks, quartz crystal clocks, biological clocks (aging), atomic clocks, radioactive clocks, and a clock in which the ticking element is a pulse of light flashing back and forth between two mirrors. Let all these clocks be adjusted to run at the same rate when at rest in the rocket frame. When these clocks fly past standard clocks in the laboratory frame, show that the phenomenon of time dilation (Ex. 10) occurs quite independently of the inner workings of the clocks. (Discussion: How does it happen that the construction of the clocks never came into discussion before? With flashes of light flying back and forth from one clock to another for purposes of synchronization, is any clock machinery really needed? Did one ever need anything more than a first light pulse, say from an electric spark, and half-silvered mirrors at measured locations here and there (Fig. 45) to create definite time delays?)

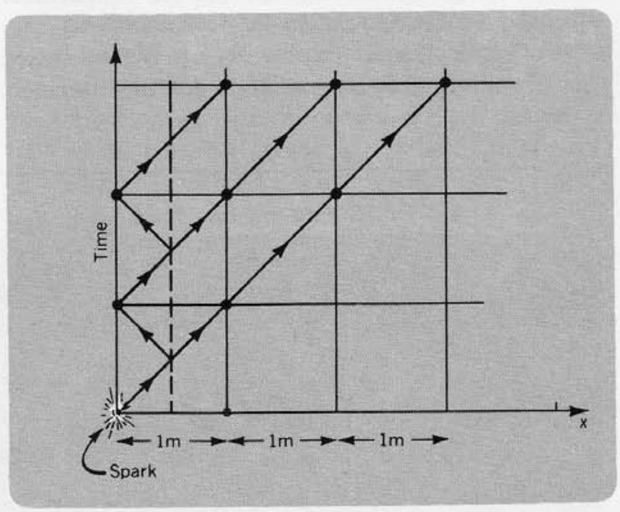


Fig. 45. Measurement of time using no clocks. Dashed line is world line of half-silvered mirror.

31. Earthbound inertial reference frames

A reference frame is inertial within some region of space and time if test particles at rest remain at rest within some specified accuracy throughout that region of spacetime. A spaceship in free fall near the earth has been shown to be effectively an inertial frame for time periods of a few seconds. Many experiments involving fast-moving particles and light itself are observed in earthbound laboratories, which are not in free fall! The force of gravity is present in an earthbound laboratory. Nevertheless, some of these experiments require so little time that a test particle released as the experiment begins does not fall very far before the experiment is over. For the purposes of many experiments, therefore, the earthbound laboratory is an inertial frame with considerable accuracy.

(a) An elementary particle with a velocity 0.96 that of light passes through a cubical spark chamber with edges 1 meter long. How far will a separate test particle released from rest fall in the gravitational field of the earth in this time? Compare your answer with the dimensions of an atomic nucleus (a few times 10^{-15} meters). Summarize by stating the dimensions of the spacetime region in which the laboratory or earthbound frame is idealized to be inertial and the specified accuracy. How big would the spark chamber have to be in order that the separate test particle would drop a *measurable* amount from rest in the time that an elementary particle of speed 0.96 c traverses the chamber?

(b) In the Michelson-Morley experiment (Ex. 33) a beam of light is reflected back and forth between pairs of mirrors about 2 meters apart so that it travels a total distance of 22 meters. How far will a test particle fall from rest in the gravitational field of the earth during the time that a particular photon traverses the Michelson-Morley equipment? To what accuracy is the earthbound frame inertial in the spacetime region in which the Michelson-Morley experiment is carried out?

32.* Size of an inertial frame

How large can a given region of space be ($\Delta x = \Delta y = \Delta z = L$, meters), how *long* can it be studied (Δt, meters!), and how *close* can it be to a center of gravitational attraction, before a detectable discrepancy, ϵ, from an ideal inertial reference system shows up?

(a) *One kind of discrepancy: relative acceleration perpendicular to the line of attraction.*

(1) Special case. Two ball bearings are released from rest from a common height of 250 meters above the earth and 25 meters apart (Fig. 46). Show that they will move closer together by a distance of about 10^{-3} meter before striking the earth. (Analyze by the method of similar triangles or by some other method. This is the example treated on page 9 of the text.) The time to fall 250 meters at an acceleration of 9.8 meters per second per second is about 7 seconds or 21×10^8 meters of light-travel time. In summary, a falling railway coach can be treated as an inertial reference system under these conditions:

ϵ (*smallest discrepancy given instruments can detect*)	*Conditions that are adequate to guarantee that discrepancy from ideal inertial frame cannot be detected* — r (*distance from center of earth*)	Δx (*horizontal spread*)	Δy *and* Δz (*spread of region in other two directions*)	Δt (*time of observation*)
$\epsilon \geq 1 \times 10^{-3}$ meter	$r \geq r_e = 6.4 \times 10^6$ meters	$\Delta x = L \leq 25$ meters	Assumed zero in analysis; therefore have to be assumed zero here in default of further analysis (part c)	$\Delta t \leq 21 \times 10^8$ meters (7 seconds)

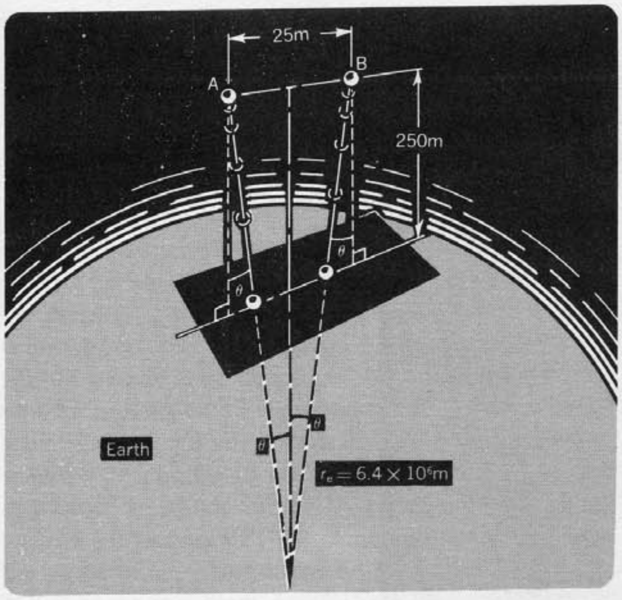


Fig. 46. Ball bearings released side by side near the earth move *closer together* as they descend. (Figure is not drawn to scale.)

(2) More general case. Test particle B is displaced Δx relative to test particle A. They have the same distance r from the center of attraction, and are studied for a time Δt. Denote by a the common acceleration of the two particles towards the center of attraction in meters per second per second, or by $a^* = a/c^2$ the value of the same acceleration measured in meters of distance per meter of time per meter of time. Show that the acceleration of particle B *relative to* A $(\Delta a^x)^*$ (meters of distance per meter of time per meter of time) is given by the formula

$$(\Delta a^x)^* = -(\Delta x/r)a^* \tag{52}$$

(Assume that the relevant angles are so small that the sine and the tangent and the angle itself can all be identified with one another.)

(b) *Another kind of discrepancy: relative acceleration parallel to the line of attraction.*

(1) General case. Test particle B is displaced by the amount Δz relative to A and parallel to r. Therefore B is further away from the center of attraction than A and experiences a smaller attraction. Consequently B is left behind A—or, as seen by an observer on A, B has a relative acceleration in the positive z direction. Show that this relative acceleration (in meters of distance per meter of time per meter of time) is

$$(\Delta a^z)^* = +(2\,\Delta z/r)\,a^* \tag{53}$$

(Hint: Use the fact that a^* falls off according to Newton's inverse square law of gravitation; thus, $a^* = \text{constant}/r^2$. Evaluate at r and at $r + \Delta z$ and take the difference. Take advantage of the fact that Δz is very small (a few meters) compared to r (thousands of kilometers) to simplify the result!)

(2) Special case. (Page 9 of text.) One test particle is 250 meters above the surface of the earth, the other 275 meters. How much will the 25 meter separation between the particles be increased in the approximately 7 seconds it takes for the first particle to hit the ground? (Hint: By what factor do the expressions for Δa^z in part b,1, and Δa^x in part a,2, differ from each other?) Use your result to complete—or, if you wish, to revise—the table in part a,1.

(c) *Case in which the region of experimentation is far from the center of the earth.*

The space corporation increases the scope of the experiments with test particles and light rays. The research group finds that the region used for previous experiments is not large enough for the new program and 7 seconds is not long enough. Management agrees to its recommendation for a space $\Delta x = 200$ meters, $\Delta y = 200$ meters, $\Delta z = 100$ meters and a time of 100 seconds, with the same tolerance as before, $\epsilon = 1 \times 10^{-3}$ meters = 1 millimeter. To how many earth radii from the center of the earth must the equipment be boosted by rockets to make the departures from ideality less than the tolerable upper limit? (Some possible questions to ask along the way: How does a^* vary with the distance r from the center of the earth? How do $(\Delta a^x)^*$ and $(\Delta a^z)^*$ vary with r? How do Δx and Δz depend upon $(\Delta a^x)^*$ and $(\Delta a^z)^*$ and the time Δt?)

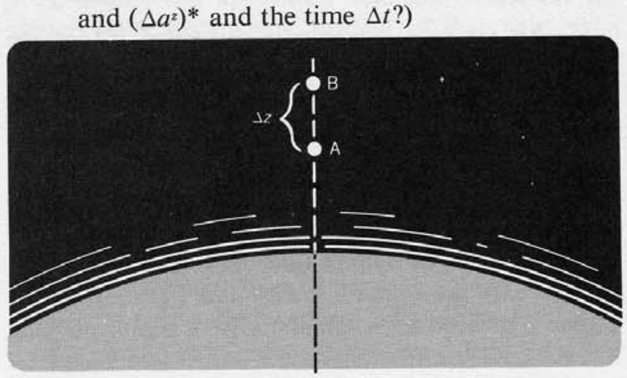


Fig. 47. Ball bearings released one above the other near the earth move *farther apart* as they descend. (Figure is not drawn to scale.)

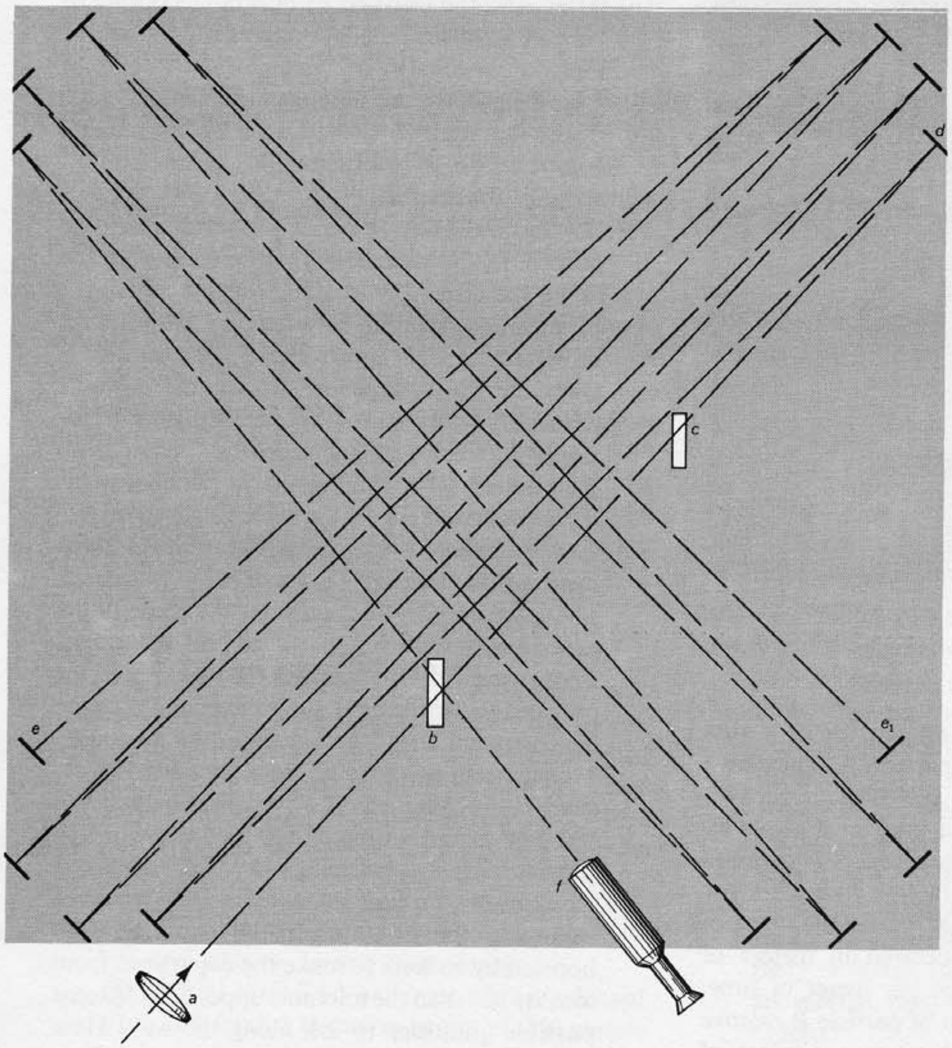


Fig. 48. Michelson-Morley interferometer mounted on a rotating marble slab.

33.* The Michelson-Morley experiment†

(a) An airplane moves with air speed c (not the speed of light) from point A to point B on the earth. A stiff wind of speed v is blowing from B toward A. Show that the time for a round trip from A to B and back to A under these circumstances is greater by a factor $1/(1 - v^2/c^2)$ than the corresponding round-trip time in still air. Paradox: The wind helps on one leg of the flight as well as hinders on the other: Why, therefore, is the round-trip time not the same in the presence of wind as in still air? Give a simple physical reason for this difference. What happens when the wind speed is nearly equal to the speed of the airplane?

(b) The same airplane now makes a round trip between A and C. The distance between A and C is the same as the distance from A to B, but the line from A to C is perpendicular to the line from A to B, so that in moving between A and C the plane flies *across* the wind. Show that the round-trip time between A and C under these circumstances is greater by a factor $1/(1 - v^2/c^2)^{1/2}$ than the corresponding round-trip time in still air.

(c) Two airplanes with the same air speed c start from A at the same time. One travels from A to B and back to A, flying first against and then with the wind (wind speed v). The other travels from A to C and back to A, flying across the wind. Which one will arrive

†A. A. Michelson and E. W. Morley, American Journal of Science, **34**, 333 (1887). The logical position of the experiment in the theory of relativity is outlined by H. P. Robertson, in Reviews of Modern Physics, **21**, 378 (1949).

home first, and what will be the difference in their arrival times? Using the binomial theorem show that if $v \ll c$, then an approximate expression for this time difference is $\Delta t = \frac{L}{2c}\left(\frac{v^2}{c^2}\right)$, where L is the round-trip distance between A and B (and between A and C).

(d) The South Pole Air Station is the supply depot for research huts on a circle of 300-kilometer radius centered on the air station. Every Monday many supply planes start simultaneously from the station and fly radially in all directions at the same altitude. Each plane drops supplies and mail to one of the research huts and flies directly home. A Fussbudget with a stopwatch stands on the hill overlooking the air station. He notices that the planes do not all return at the same time. This discrepancy perplexes him because he knows from careful measurement that (1) the distance from the air station to every research hut is the same, (2) every plane flies with the same air speed as every other plane—300 kilometers per hour, and (3) every plane travels in a straight line over the ground from station to hut and back. The Fussbudget finally decides that the discrepancy is due to the wind at the high altitude at which the planes fly. With his stopwatch he measures the time from the return of the first plane to the return of the last plane to be 4 seconds. What is the wind speed at the altitude where the planes fly? What can the Fussbudget say about the *direction* of this wind?

(e) In their famous experiment Michelson and Morley attempted to detect the so-called *ether drift*—the motion of the earth through the "ether," with respect to which light was supposed to have the velocity c. They compared the round-trip times for light to travel equal distances parallel and perpendicular to the direction of motion of the earth around the sun. They reflected the light back and forth between nearly parallel mirrors. (This would correspond to part c if each airplane made repeated round trips.) By this means they were able to use a total round-trip length of 22 meters for each path. If the "ether" is at rest with respect to the sun, and if the earth moves at 30×10^3 meters per second in its path around the sun, what is the approximate difference in time of return between light flashes that are emitted simultaneously and travel along the two perpendicular paths? Even with the instruments of today, the difference predicted by the ether-drift hypothesis would be too small to measure directly, and the following method was used instead.

(f) The original Michelson-Morley *interferometer* is diagramed in Fig. 48. Nearly monochromatic light (light of a single frequency) enters through the lens at a. Some of the light is reflected by the half-silvered mirror at b and the rest of the light continues toward d. Both beams are reflected back and forth until they reach mirrors e and e_1 respectively, where each beam is reflected back upon itself and retraces its path to mirror b. At mirror b parts of each beam combine to enter telescope f together. The transparent piece of glass at c, of the same dimensions as the half-silvered mirror b, is inserted so that both beams pass the same number of times (three times) through this thickness of glass on their way to telescope f.

Suppose that the perpendicular path lengths are exactly equal and the instrument is at rest with respect to the ether. Then monochromatic light from the two paths that leaves mirror b in some relative phase will return to mirror b in the same phase. Under these circumstances the waves entering telescope f will *add* and the image in this telescope will be *bright*. On the other hand, if one of the beams has been delayed a time corresponding to one-half period of the light, then it will arrive at mirror b one-half period later, and the waves entering the telescope will *cancel*, so the image in the telescope will be *dark*. If one beam is retarded a time corresponding to one whole period, the telescope image will be bright, and so forth. What time interval corresponds to one period of the light? Michelson and Morley used sodium light of wavelength 5890 angstroms (one angstrom is equal to 10^{-10} meter). From the equations $\nu\lambda = c$ and $\nu = 1/T$ show that one period of sodium light corresponds to about 2×10^{-15} seconds.

Now there is no way to "turn off" the alleged ether drift, adjust the apparatus, and then turn on the alleged ether drift again. Instead of this, Michelson and Morley floated their interferometer in a pool of mercury and rotated it slowly about its center like a phonograph record while observing the image in the telescope (Fig. 48). In this way if light is delayed on either path when the instrument is oriented in a certain direction, light on the *other* path will be delayed by the same amount of time when the instrument has rotated 90 degrees. Hence the *total change* in delay time between the two paths observed as the interferometer rotates should be *twice* the difference calculated using the expression derived in part c.

By simple refinements of this method Michelson and Morley were able to show that the time change between the two paths as the instrument rotated corresponded to less than *one one-hundredth* of the shift from one dark image in the telescope to the next dark image. Show that this result implies that the motion of the ether at the surface of the earth—if it exists at all—is less than one sixth of the speed of the earth in its orbit. In order to eliminate the possibility that the ether was flowing past the sun at the same rate as the earth was moving its orbit, they repeated the experiment at intervals of three months, always with negative results.

(g) Does the Michelson-Morley experiment, by itself, disprove the theory that light is propagated through an ether? Can the ether theory be modified to agree with the results of this experiment? How? What further experiment can be used to test the modified theory?

34.* The Kennedy-Thorndike experiment†

The Michelson-Morley experiment was designed to detect any motion of the earth relative to a hypothetical fluid—the ether—a medium in which light was supposed to move with characteristic speed c. No such relative motion of earth and ether was detected. Partly as a result of this experiment the concept of ether has since been discarded. In the modern view, light requires no medium for its transmission.

What significance does the negative result of the Michelson-Morley experiment have for us who do not believe in the ether theory of light propagation? Simply this: (1) The round-trip speed of light measured on earth is the same in every direction—the speed of light is *isotropic*. (2) The speed of light is isotropic not only when the earth moves in one direction around the sun in, say, January (call the earth with this motion the "laboratory frame"); but also when the earth moves in the *opposite* direction around the sun six months later, in July (call the earth with this motion the "rocket frame"). (3) The generalization of this result to any pair of inertial frames in relative motion is contained in the statement, *the round-trip speed of light is isotropic both in the laboratory frame and in the rocket frame.*

This result leaves an important question unanswered: Does the round-trip speed of light—which is isotropic in both laboratory and rocket frames—also have *the same numerical value* in laboratory and rocket frames? The assumption that this speed has the same numerical value in both frames played a central role in demonstrating the invariance of the interval (Section 5). But is this assumption valid?

(a) An experiment to test the assumption of the equality of the round-trip speed of light in two inertial frames in relative motion was conducted in 1932 by Roy J. Kennedy and Edward M. Thorndike. The experiment uses an interferometer with arms of *unequal* length (Fig. 49). Assume that one arm of the interferometer is Δl longer than the other arm. Show that a flash of light entering the apparatus will take a time $2\Delta l/c$ longer to complete the round trip along the longer arm than along the shorter arm. The difference in length Δl used by Kennedy and Thorndike was approximately 16 centimeters. What is the approximate difference in time for the round trip of a light flash along the alternative paths?

(b) Instead of a pulse of light, Kennedy and Thorndike used continuous monochromatic light of period $T = 1.820 \times 10^{-15}$ seconds ($\lambda = 5461$ angstroms) from a mercury source. Light that traverses the longer arm of the interferometer will return approximately how many periods n later than light that traverses the shorter arm? If in the actual experiment the number of periods is an integer, the reunited light from the two arms will *add* and the field of view seen through the telescope will be *bright*. In contrast, if in the actual experiment the number of periods is a half-integer, the reunited light from the two arms will *cancel* and the field of view of the telescope will be *dark*.

(c) The earth continues on its path around the sun. Six months later the earth has reversed the direction of its velocity relative to the fixed stars. In this new frame of reference will the round-trip speed of light have the same numerical value c as in the original frame of reference? One can rewrite the answer to part b for the original frame of reference in the form

$$c = (2/n)(\Delta l/T) \tag{54}$$

where Δl is the difference in length between the two interferometer arms, T is the time for one period of the atomic light source, and n is the number of periods that elapse between the return of the light on the shorter path and the return of the light on the longer path. Suppose that as the earth orbits the sun no shift is observed in the telescope field of view from, say, light toward dark. This means that n is observed to be constant. What would this hypothetical result tell about the numerical value c of the speed of light? Point out the *standards* of distance and time used in determining this result, as they appear in Eq. 54. Quartz has the greatest stability of dimension of any known material. Atomic time standards have proved to be

†The report of the original experiment is found in the Physical Review, **42**, 400, (1932). The logical position of the experiment in the theory of relativity is outlined by H. P. Robertson in the Reviews of Modern Physics, **21**, 378 (1949).

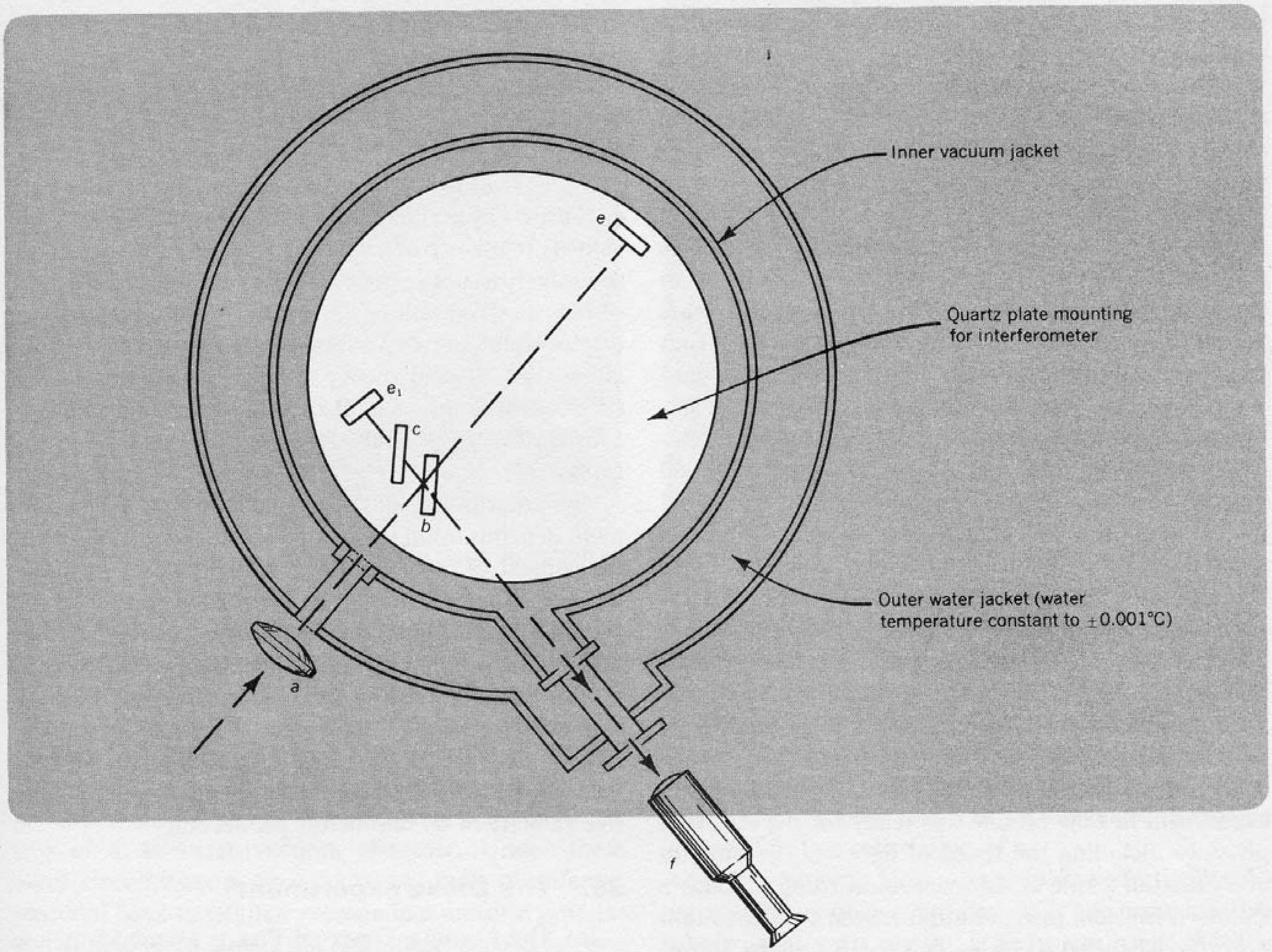


Fig. 49. Schematic diagram of apparatus used for the Kennedy-Thorndike experiment. Parts of the interferometer have been labeled with letters corresponding to those used in describing the Michelson-Morley interferometer (Ex. 33). The experimenters went to great lengths to insure the optical and mechanical stability of their apparatus. The interferometer is mounted on a plate of quartz, which changes dimension very little when temperature changes. The interferometer is enclosed in a vacuum jacket so that changes in atmospheric pressure will not alter the effective optical path length of the interferometer arms (slightly different speed of light at different atmospheric pressure!). The inner vacuum jacket is surrounded by an outer water jacket in which the water is kept at a temperature that varies less than ± 0.001 degree centigrade. The entire apparatus shown in the figure is enclosed in a small darkroom (not shown) maintained at a temperature constant within a few hundredths of a degree. The small darkroom is in turn enclosed in a larger darkroom whose temperature is constant within a few tenths of a degree. The overall size of the apparatus can be judged from the fact that the difference in length of the two arms of the interferometer (length *be* compared with length be_1) is 16 centimeters.

the most dependable earth-bound timekeeping mechanisms.

(d) In order to carry out the experiment outlined in the preceding paragraphs, Kennedy and Thorndike would have had to keep their interferometer operating perfectly for half a year while continuously observing the field of view through the telescope. Uninterrupted operation for so long a time was not feasible. The actual durations of their observations varied from eight days to a month. There were several such periods of observation at three-month intervals. From the data obtained in these periods, Kennedy and Thorndike were able to estimate that over a single six-month observation the number of periods n of relative delay would vary by less than the fraction 3/1000 of one period. Take the differential of Eq. 54 to find the largest fractional change dc/c of the round-trip speed of light between the two frames consistent with this estimated change in n (frame No. 1—the "laboratory" frame—and frame No. 2—the "rocket" frame—being in the present analysis the earth itself at two different times of year, with a relative velocity twice the speed of the earth in its orbit: 2×30 kilometers per second).

Historical note: At the time of the Michelson-Morley experiment in 1887 no one was ready for the idea that physics—including the speed of light—is the same in every inertial frame of reference. According to today's standard Einstein interpretation it seems obvious that both the Michelson-Morley and the Kennedy-Thorndike experiments should give null results. However, when Kennedy and Thorndike made their measurements in 1932, two alternatives to the Einstein theory were open to consideration (designated here as theory A and theory B). Both A and B assumed the old idea of an absolute space, or "ether," in which light has the speed c. Both A and B explained the zero fringe shift in the Michelson-Morley experiment by saying that all matter that moves at a velocity v relative to "absolute space" undergoes a shrinkage of its space dimensions in the direction of motion to a new length equal to $(1 - v^2/c^2)^{1/2}$ times the old length ("Lorentz-FitzGerald contraction hypothesis"). The two theories differed as to the effect of "motion through absolute space" on the running rate of a clock. Theory A said, no effect. Theory B said that a standard seconds clock moving through absolute space at velocity v has a time between ticks of $(1 - v^2/c^2)^{1/2}$ seconds. On theory B the ratio $\Delta l/T$ in Eq. 54 will not be affected by the velocity of the clock, and the Kennedy-Thorndike experiment will give a null result, as observed ("complicated explanation for simple effect"). On theory A the ratio $\Delta l/T$ in Eq. 54 will be multiplied by the factor $(1 - v_1^2/c^2)^{1/2}$ at a time of year when the "velocity of the earth relative to absolute space" is v_1; and multiplied by $(1 - v_2^2/c^2)^{1/2}$ at a time of year when this velocity is v_2. Thus the fringes should shift from one time of year ($v_1 = v_{\text{orbital}} + v_{\text{sun}}$) to another time of year ($v_2 = v_{\text{orbital}} - v_{\text{sun}}$) unless by accident the sun happened to have "zero velocity relative to absolute space"—an accident judged so unlikely as not to provide an acceptable explanation of the observed null effect. Thus the Kennedy-Thorndike experiment ruled out theory A (length contraction alone) but allowed theory B (length contraction plus time contraction)—and also allowed the much simpler Einstein theory of equivalence of all inertial reference frames.

The "sensitivity" of the Kennedy-Thorndike experiment depends upon the theory under consideration. In the context of theory A the observations set an upper limit of about 15 kilometers per second to the "speed of the sun through absolute space" (sensitivity reported in the Kennedy-Thorndike paper). In the context of Einstein's theory the observations say that the round-trip speed of light has the same numerical magnitude—within an error of about 2 meters per second—in inertial frames of reference having a relative velocity of 60 kilometers per second.

35.* The Dicke experiment†

(a) The Leaning Tower of Pisa is about 55 meters high. Galileo says, "the variation of speed in air between balls of gold, lead, copper, porphyry, and other heavy materials is so slight that in a fall of 100 cubits [about 46 meters] a ball of gold would surely not outstrip one of copper by as much as four fingers. Having observed this I came to the conclusion that in a medium totally devoid of resistance all bodies would fall with the same speed."‡ Taking four fingers to be equal to seven centimeters, find the *maximum fractional difference* in the acceleration of gravity $\Delta g/g$ between balls of gold and copper that would be con-

†R. H. Dicke, "The Eötvös Experiment," Scientific American, **205,** 84 (December, 1961). See also P. G. Roll, R. Krotkov, and R. H. Dicke, Annals of Physics, **26,** 442 (1964). The first of these articles is a popular exposition written early in the course of the present experiment. The second article reports the final results of the experiment and takes on added interest because of its account of the elaborate precautions required to insure that no influence that might affect the experiment was disregarded.

‡Galileo Galilei, *Dialogues Concerning Two New Sciences*, translated by Henry Crew and Alfonso de Salvio (Northwestern University Press, Evanston, Illinois, 1950).