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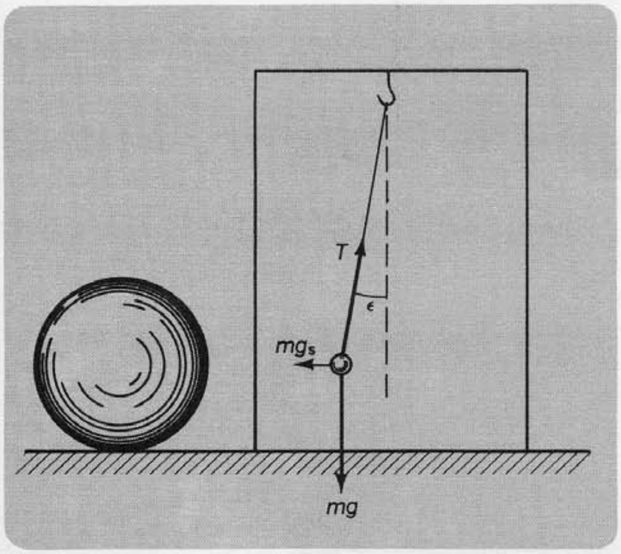


Fig. 50. Nearby massive sphere results in static deflection of plumb line from vertical.

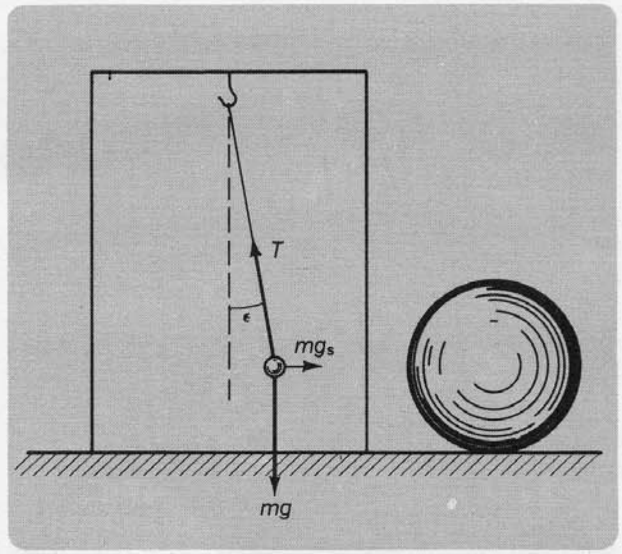


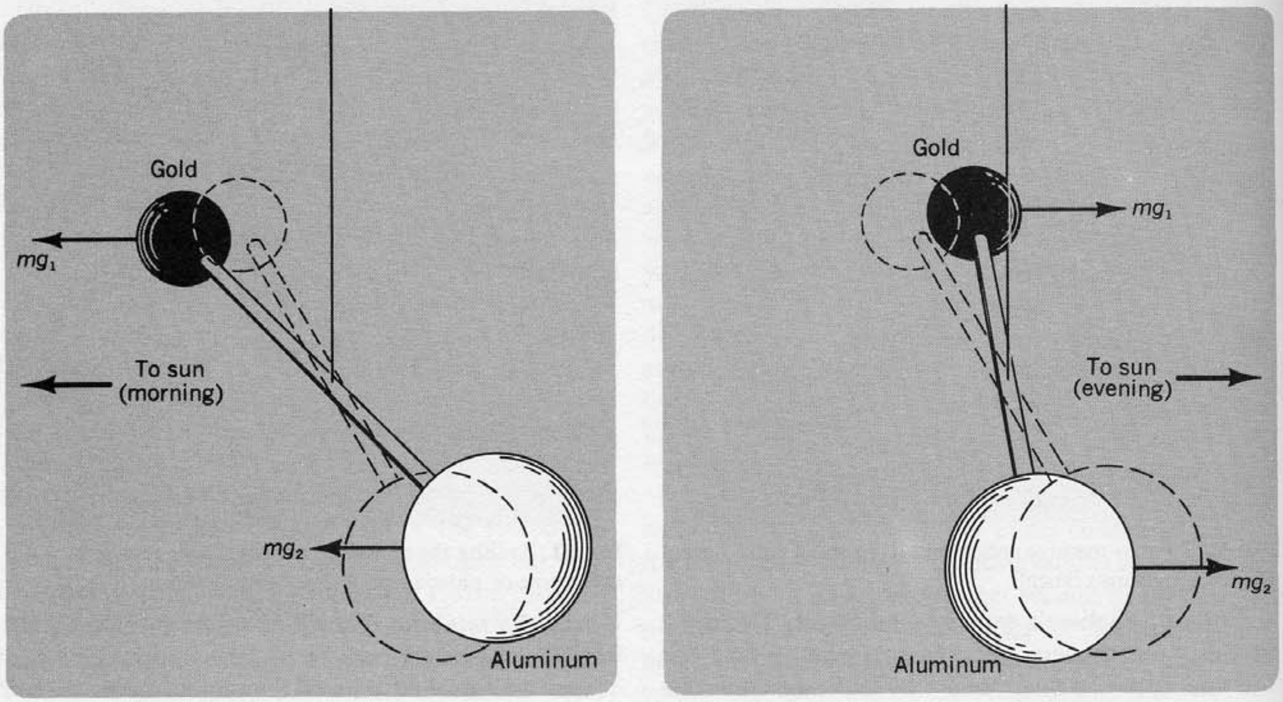
Fig. 51. Rolling the sphere to the other side results in static deflection of plumb line in opposite direction.

sistent with Galileo's experimental result. The result of the more modern Dicke experiment is that this fraction is not greater than 3×10^{-11}. Assume that the fraction has the more recently determined maximum value. Determine how far behind the first ball the second one will be when the first reaches the ground if they are dropped simultaneously from the top of a 46-meter vacuum chamber. Under these same circumstances, how far would balls of different material have to fall in a vacuum in a uniform gravitational field of 10 meters per second per second for one ball to lag behind the other one by a distance of 1 millimeter? Compare this distance with that of the moon from the earth (3.8×10^8 meters). Clearly the Dicke experiment was not carried out using falling balls!

(b) A plumb bob of mass m hangs on the end of a long line from the ceiling of a closed room (Fig. 50). A very massive sphere at one side of the closed room exerts a horizontal gravitational force mg_s on the plumb bob, where $g_s = GM/R^2$, M being the mass of the large sphere, and R the distance between plumb bob and the center of the sphere. This horizontal force causes a static deflection of the plumb line from the vertical by the small angle ϵ. (Similar practical example: In northern India the mass of the Himalaya Mountains results in a slight sideways deflection of plumb lines, causing difficulties in precise surveying!) The sphere is now rolled around to a corresponding position on the other side of the room (Fig. 51) causing a static deflection of the plumb by an angle ϵ of the same magnitude but in the opposite direction. Now the angle ϵ is very small. (Deflection due to the Himalayas is about 5 seconds of arc, which equals 0.0014 degrees!) However, as the sphere is rolled around and around outside the closed room, an observer inside the room can measure the gravitational field g_s due to the sphere by measuring with greater and greater precision the total deflection angle $2\epsilon \approx 2 \sin \epsilon$ of the plumb line. Derive the equation that he will need in this calculation of g_s.

(c) We on earth have a large sphere effectively rolling around us once every day. It is the most massive sphere in the solar system: it is the sun itself! What is the value of the gravitational acceleration $g_s = GM/R^2$ due to the sun at the position of the earth? (Some constants useful in this calculation appear inside the front cover of this book.)

(d) One additional acceleration must be considered that, however, will not enter our final comparison of gravitational acceleration g_s for different materials. This additional acceleration is the centrifugal acceleration due to the motion of the earth around the sun. When you round a corner in a car you are pressed against the side of the car on the outward side of the turn. This outward force—called the *centrifugal pseudo-force* or the *centrifugal inertial force*—is due to the acceleration of your reference frame (the car) toward the center of the circular turn. This centrifugal inertial force has the value mv^2/r where v is the speed of the car and r is the radius of the turn. Now the earth moves around the sun in a path that is nearly circular. The sun's gravitational force mg_s acts on a plumb bob in a direction *toward* the sun; the centrifugal inertial force mv^2/R acts in a direction *away* from the sun. Compare the "centrifugal acceleration" v^2/R at the position of



A. Hypothetical effect: morning

B. Hypothetical effect: evening

Fig. 52. Schematic diagram of the Dicke experiment. Any difference in the gravitational acceleration of the sun for gold and aluminum should result in opposite sense of net torque on torsion pendulum in the evening compared with the morning. Large aluminum ball has the same mass as small high-density gold ball.

the earth with the oppositely directed gravitational acceleration g_s calculated in part c. What is the net acceleration toward or away from the sun of a particle riding on the earth as observed in the (accelerated) frame of the earth?

(e) Of what use is the discussion thus far? A plumb bob hung near the surface of the earth experiences a gravitational acceleration g_s toward the sun—and an equal-but-opposite centrifugal acceleration v^2/R away from the sun. Therefore—in the accelerating reference frame of the earth—the bob experiences no net force at all due to the presence of the sun. Indeed this is the method by which we constructed an inertial frame in the first place (Section 2): Let the frame be in free fall about the center of gravitational attraction. A particle at rest on the earth's surface is in free fall about the sun and therefore experiences no net force due to the sun. What then does all this have to do with measuring the equality of gravitational acceleration for particles made of different substances—the subject of the Dicke experiment? Answer: Our purpose is to detect the difference—if any—in the gravitational acceleration g_s toward the sun for different materials. The centrifugal acceleration v^2/R away from the sun is presumably the same for all materials and therefore need not enter any *comparison* of different materials. Consider a torsion pendulum suspended from its center by a thin quartz fiber (Fig. 52,A). A light rod of length l supports at its ends two bobs of equal mass made of different materials—say aluminum and gold. *Suppose* that the gravitational acceleration g_1 of the gold due to the sun is slightly greater than the acceleration g_2 of the aluminum due to the sun. Then there will be a slight net torque on the torsion pendulum due to the sun. For the position of the sun shown in Fig. 52,A, show that the net torque is *counterclockwise* when viewed from above. Show also that the magnitude of this net torque is given by the expression

$$\text{torque} = mg_1\frac{l}{2} - mg_2\frac{l}{2} = m(g_1 - g_2)\frac{l}{2} = mg_s\left(\frac{\Delta g}{g_s}\right)\frac{l}{2} \tag{55}$$

Suppose that the fraction $(\Delta g/g_s)$ has the maximum value, 3×10^{-11}, consistent with the results of the final experiment, that l has the value 0.06 meters, and that each bob has a mass of 0.03 kilograms. What is the magnitude of the net torque? Compare this to the torque provided by the added weight of a bacterium of mass 10^{-15} kilogram placed on the end of a meter stick balanced at its center in the gravitational field of the earth.

(f) The sun moves around the heavens as seen from the earth. Twelve hours later the sun is located as shown in Fig. 52,B. Show that under these changed circumstances the net torque will have the same magnitude as that calculated above but now will be *clockwise* as viewed from above—in a sense opposite to that of part e! This change in the sense of the torque every twelve hours allows a small difference $\Delta g = g_1 - g_2$ in the acceleration of gold and aluminum to be detected using the torsion pendulum. As the torsion pendulum jiggles on its fiber because of random motion, passing trucks, earth tremors and so forth, one needs to consider only those deflections that keep step with the changing position of the sun.

(g) A torque on the rod causes an angular rotation of the quartz fiber of θ radians given by the formula

$$\text{torque} = k\theta$$

where k is called the *torsion constant* of the fiber. Show that the maximum angular rotation of the torsion pendulum from one side to the other during one rotation of the earth is given by the expression

$$\theta_{\text{tot}} = \frac{mg_s l}{k}\left(\frac{\Delta g}{g}\right)$$

(h) In practice Dicke's torsion balance can be thought of as consisting of 0.030 kilogram gold and aluminum bobs mounted on the ends of a beam 6×10^{-2} meter in length suspended in a vacuum on a quartz fiber of torsion constant 2×10^{-8} newton meter per radian. A statistical analysis of the angular displacements of this torsion pendulum over long periods of time leads to the conclusion that the fraction $\Delta g/g$ for gold and aluminum is less than 3×10^{-11}. To what mean maximum angle of rotation from side to side during one rotation of the earth does this correspond? Random motions of the torsion pendulum—noise!—are of much greater amplitude than this; hence the need for the statistical analysis of the results using a programmed computer.

36.* Down with relativity!

Mr. Van Dam is an intelligent and reasonable man with a knowledge of high school physics. He has the following objections to the theory of relativity. Answer each of Mr. Van Dam's objections decisively—*without criticizing him!* If you wish, you may present a single connected account of how and why one is driven to relativity, in which these objections are all answered.

(a) "A says B's clock goes slow, and B says that A's clock goes slow. This is a logical contradiction. Therefore relativity should be abandoned."

(b) "A says B's meter sticks are contracted, and B says A's meter sticks are contracted. This is a logical contradiction. Therefore relativity should be abandoned."

(c) "Relativity does not even have a *unique* way to *define* space and time coordinates. Therefore anything it says about velocities (and hence about motions) is without meaning."

(d) "Relativity postulates that light travels with a standard speed regardless of the inertial frame from which its progress is measured. This postulate is certainly wrong. Anybody with common sense knows that travel at high speed in the direction of a receding light pulse will decrease the speed with which the pulse recedes. Hence light *cannot* have the same speed for observers in relative motion. With this disproof of the basic postulate all of relativity collapses."

(e) "There isn't a single experimental test of the *results* of special relativity."

(f) "Relativity offers no way to describe an event without coordinates—and no way to speak about *coordinates* without referring to one or another particular *reference frame*. However, physical events have an existence *independent* of all choice of coordinates and all choice of reference frame. Hence relativity—with its coordinates and reference frames—cannot provide a valid description of these events."

(g) "Relativity is concerned only with how we *observe* things, not what is *really* happening. Hence it is not a scientific theory, since science deals with *reality*."

E. APPROXIMATIONS AT LOW VELOCITY

37. Euclidean analogy—a worked example

There is a very small angle, θ_r, between the respective axes of two rotated Euclidean frames. Use the series expansions of Table 8 to find an approximate set of transformation equations between the coordinates of a given point with respect to two reference frames. Neglect powers of θ_r higher than the first.

Solution: From Table 8, for small θ_r

$$\sin\theta_r \approx \theta_r$$
$$\cos\theta_r \approx 1$$

Therefore the Euclidean transformation equations (inverse of Eqs. 29) become

$$x' = x \cos \theta_r - y \sin \theta_r \approx x - \theta_r y \qquad (56)$$
$$y' = x \sin \theta_r + y \cos \theta_r \approx \theta_r x + y$$

This approximate transformation can be made as accurate as desired by making θ_r sufficiently small.

38. The Galilean transformation

Suppose that β_r is very small. Then $\beta_r = \tanh \theta_r \approx \theta_r$. Use the series expansions of Table 8 to show that if terms that contain powers of θ_r higher than the first are neglected, the transformation equations become

$$x' = x - \beta_r t \qquad (57)$$
$$(\beta_r \ll 1)$$
$$t' = -\beta_r x + t \qquad (58)$$

Now use everyday, nonrelativistic Newtonian arguments to derive the transformation equations between two reference frames. These are called the *Galilean transformation equations*

$$x' = x - v_r t_{\text{sec}} \qquad (59)$$
$$\text{(Galilean transformation)}$$
$$t'_{\text{sec}} = t_{\text{sec}} \qquad (60)$$

where v_r is the relative speed between the two frames in meters per second.

Transformation equations 57 and 58 appear to be completely inconsistent with Eqs. 59 and 60. Is this first impression *correct*, and if not, why not? (Discussion: Why does v_r in the Galilean transformation (Eq. 59) replace β_r in Eq. 57? How does Eq. 58 look when rewritten in terms of v_r and t_{sec}? How do everyday velocities compare with the speed of light?)

39.* Limits of accuracy of a Galilean transformation

Make a more accurate approximation of the transformation equations at low relative velocities by allowing terms in θ_r^2 to remain but, again, neglecting terms with higher powers of θ_r. (This is called a *second order approximation* in θ_r. Notice from the series expansion of tanh θ in Table 8 that even to second order in θ_r, $\beta_r \approx \theta_r$.) Show that the coefficients for x and t in Eqs. 57 and 58 agree with the improved second-order approximation to better than 1 percent for velocities β_r less than 1/7.

If a sports car can accelerate uniformly from rest to 60 miles per hour (about 27 meters per second) in 7 seconds, roughly how many days would it take to reach $\beta = 1/7$ at the same constant acceleration? How many days would be required to reach this speed at the greatest acceleration that the human body can stand for reasonable periods (about 7 g, or 7 times the acceleration of gravity)?

40.* Collisions Newtonian and relativistic —and the domain within which the two predictions agree to one percent

Proton A collides elastically with proton B, which is initially at rest. The outcome of the collision cannot be predicted. It depends upon the closeness of the encounter. In most events proton A will be deviated by only a slight angle α_A from its original direction of motion. Then proton B will be given only a slight kick off to the side, at an angle α_B (relative to the forward direction) that is close to 90 degrees. Occasionally there is a very close encounter in which B acquires nearly all the energy and goes off at a very small angle α_B to the forward direction. Between these two ex-

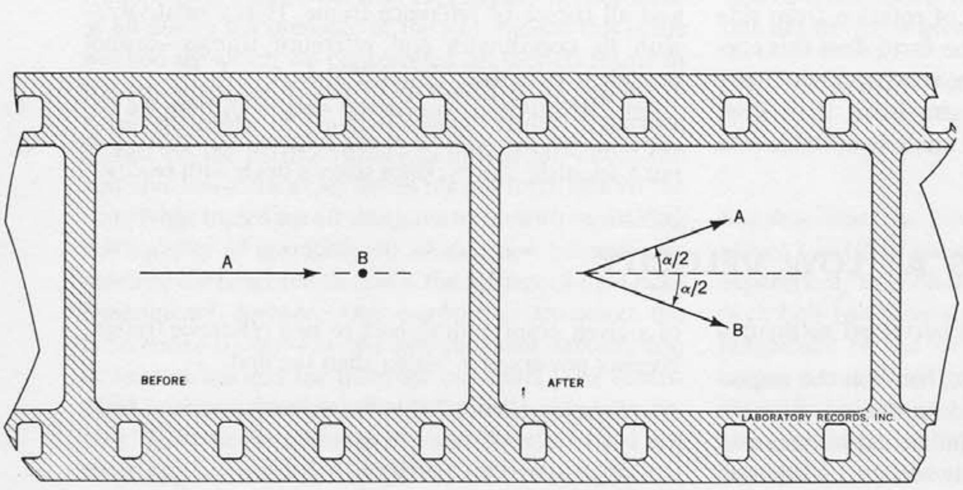


Fig. 53. Laboratory frame record of a symmetric elastic collision.

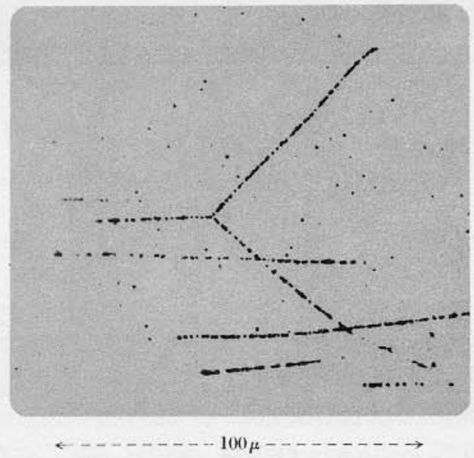


Fig. 54, A. Photograph of a nonrelativistic symmetric elastic collision between a moving proton and a second proton initially at rest. Initial speed of the incident proton is about $\beta = 0.1$. The angle between outgoing protons is 90 degrees, as predicted by Newtonian mechanics. From C. F. Powell and G. P. S. Occhialini, *Nuclear Physics in Photographs* (Oxford University Press, Oxford, 1947).

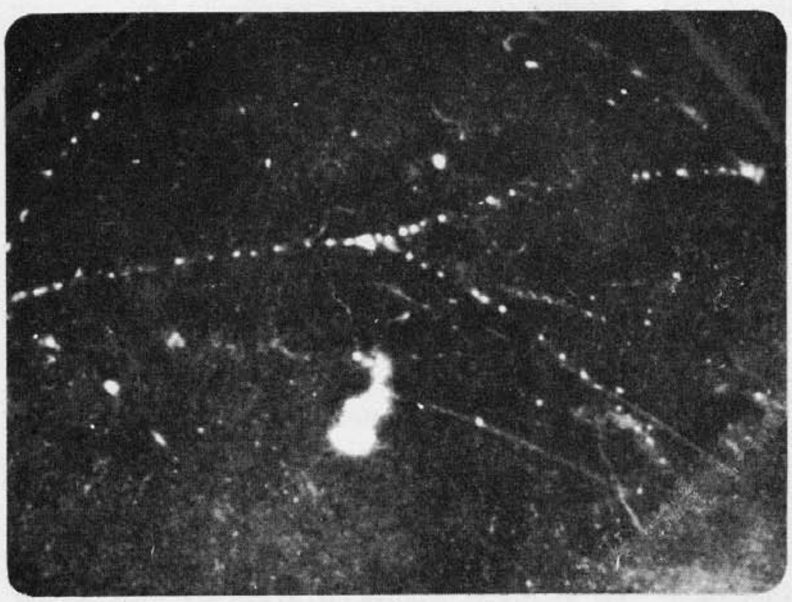


Fig. 54, B. Expansion chamber photograph of a relativistic and approximately symmetric elastic collision between a moving electron and a second electron initially at rest. Initial speed of the incident electron is about $\beta = 0.97$. The angle between outgoing electrons is much less than the 90 degree angle predicted by Newtonian mechanics. The curved path of the charged electrons is due to the presence of a magnetic field used to determine the momentum of each electron. Document Hermann Publishers, Paris.

tremes there occurs from time to time a "symmetric collision" in which the two identical particles come off with identical speeds along paths that make identical angles $\alpha_A = \alpha_B = \alpha/2$ with the forward direction (Fig. 53). *Question: How great is the angle of deflection in a symmetric collision? Discussion:* According to *Newtonian mechanics* the total angle of separation, is 90 degrees in every elastic collision (symmetric or not!). *That this angle will be less than 90 degrees for a fast impact is one of the most interesting and decisive predictions of relativity.* Figure 54,A, shows a low-velocity collision whose 90 degree angle of separation satisfies the Newtonian prediction. In contrast, Fig. 54,B, shows a high-velocity collision whose angle of separation is decisively less than 90 degrees. This circumstance means that *the difference between the separation angle from 90 degrees provides a useful measure of the departure from Newtonian mechanics.* For example, ask this question: How high must the velocity in such collision experiments be before the separation angle deviates from 90 degrees by as much as 1/100 of a radian? It greatly simplifies the analysis of this question to look at the symmetric collision pictured above from a frame of reference so chosen that one can capitalize on *symmetry arguments.* For this purpose climb onto a rocket and travel to the right with a velocity just great enough to keep up with the forward velocities of A and B after the collision. Viewed from this rocket, particles A and B therefore have *no forward velocity component:*

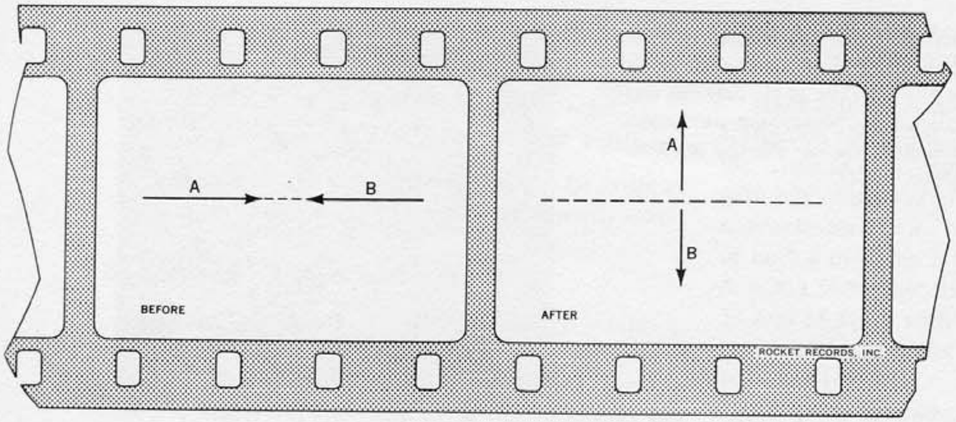


Fig. 55. Rocket frame records of the symmetric elastic collision of Fig. 53. Rocket frame is so chosen that particles A and B have no forward velocity component after the collision.

As to the lateral (up-down) velocity components of A and B, note that these were equal in magnitude and opposite in direction in the laboratory frame. Moreover, this symmetry feature of the velocity diagram cannot be altered by viewing the collision from a rocket frame moving to the right. Therefore the velocities of A and B after the collision, as viewed in the rocket frame, are equal and opposite. This conclusion is payoff No. 1 from arguments based on *symmetry*. Now for payoff No. 2—again achieved by viewing the collision in the rocket frame of reference: In this frame, and *before* the collision, A and B have velocities that are equal in magnitude and opposite in direction. Why? What inconsistency would result if these speeds were *not* equal? *Symmetry* would be violated, as one can see in the following way.

The diagram of the velocity in the rocket frame *after* the collision has *left-right symmetry*. In other words, by looking at the particles separating after the collision it is impossible to tell from what directions the particles arrived at the point of collision.

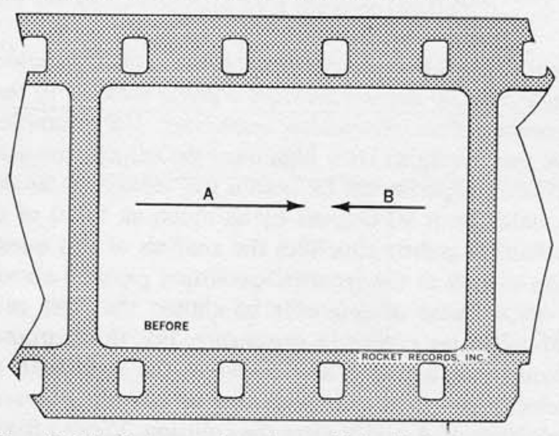


Fig. 56. Rocket record as it would be if, before the collision, particles A and B have different speeds: an incorrect assumption.

Instead of A coming from the left and B coming from the right, A could as well be coming from the right and B from the left (for example, if the viewer went around in back and looked at the collision from the other side).

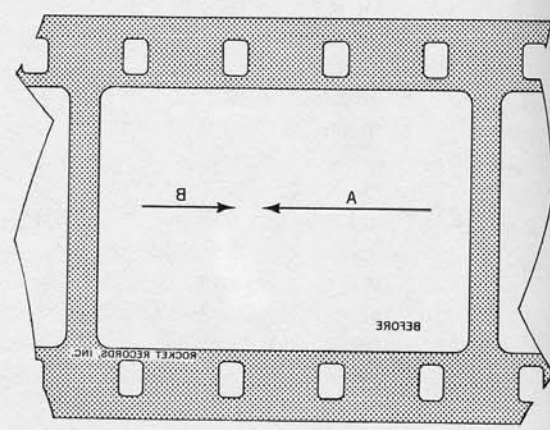


Fig. 57. Rocket record of Fig. 56 looked at from the other side.

But the colliding particles are identical—what is called B in the diagram above could as well have been called A, and conversely:

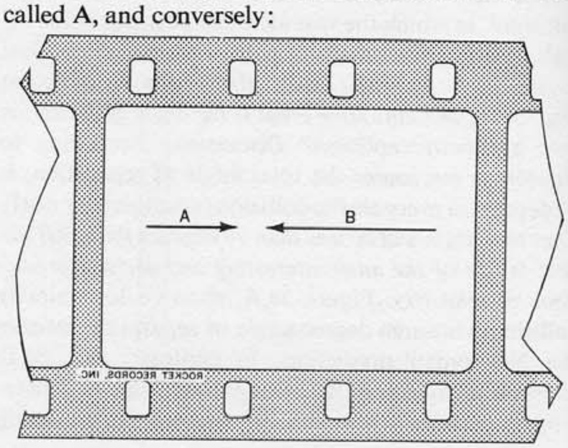


Fig. 58. Rocket record of Fig. 57 with labels A and B for identical balls interchanged.

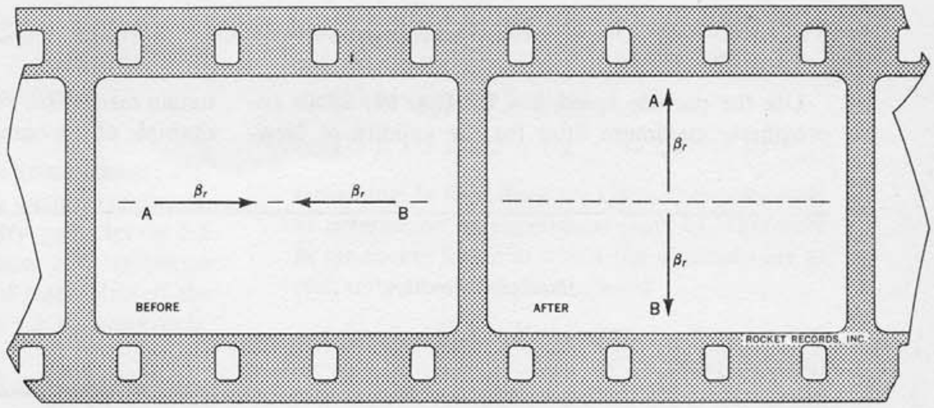


Fig. 59. Conclusion of symmetry arguments: In the rocket frame in which balls A and B have no forward velocity component after the collision, all speeds before the collision and all speeds after the collision have the same value.

Now note that we have in Figs. 56 and 58 two different initial conditions that result in one and the same outcome (Fig. 53). Moreover, these initial conditions differ only in that a suitable increase in the speed of the observing rocket transforms Fig. 56 into the appearance of Fig. 58. But the *outcome* of Fig. 56 cannot continue to look the same as the outcome of Fig. 58 after this increase in the speed of the observer. There is therefore an *inconsistency* in assuming that Fig. 56 and Fig. 58 were different in the first place. To avoid this inconsistency one must conclude that in the rocket frame *A and B have the same speed before the collision*, as drawn in Fig. 55.

Not only do A and B have equal speeds in the rocket frame before the collision—and equal speeds after the collision—but also these speeds before and after the collision are the *same*. If they were not, the following difficulty would arise. (Third use of a symmetry argument—here not symmetry in space but symmetry in time!) Make a moving picture of the collision, develop and print it, and run it *backwards* through the projector. If originally the particles *lost* speed in the collision, they will now be seen to *gain* speed. Such a difference between the two directions of time is a characteristic feature of so-called *irreversible processes*, such as (1) the flow of heat from a hotter object to a cooler one, (2) the aging of an animal, (3) the breaking of an egg, and (4) an inelastic encounter. However, we have limited attention here to *elastic* collisions. Therefore we now accept for study only those events that are *reversible* according to the following definition:

A *reversible* process is one in which it is impossible to distinguish one direction of time from the other by a difference between a film of the process run through the projector in one direction and the same film run through the projector in the other direction.

Because the collision of the two protons is *elastic*, all four speeds in Fig. 59 are *identical*.

This result is very compact and simple. The reasoning leading up to this result can be summarized in a form equally compact and simple. Merely cite these two words: "By symmetry!" Symmetry reasoning of this kind simplifies the analysis of a great variety of physical problems.

The reasoning so far, being based as it is on symmetry considerations, is the same in Newtonian and in relativistic mechanics. The difference between the accounts appears when the now completed rocket-velocity diagram is transformed back to the laboratory frame. In Newtonian mechanics velocities *add as vectors*. Therefore we have only to add the horizontal velocity β_r of the rocket frame after the collision to find the velocities of A and B in the laboratory frame after the collision:

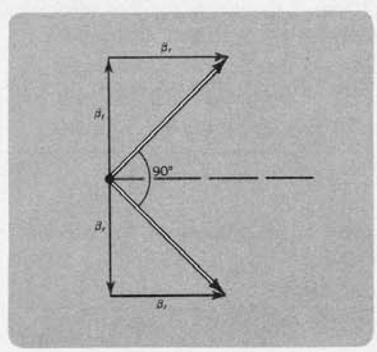


Fig. 60. Newtonian (nonrelativistic) analysis of resultant velocities in the laboratory frame after the collision.

Evidently the angle of separation α is indeed always 90 degrees in Newtonian mechanics, independent of the velocity of the original impact. Not so in relativity!

Show that the incident proton can have a velocity as great as $\beta = 2/7$ without making the angle between v_A and v_B in a symmetric collision depart from the Newtonian value of 90 degrees by as much as 0.01 radian—that is, show that Newtonian mechanics gives good accuracy for a particle with $(2/7)c$ colliding with a particle at rest (or particles with velocity $(1/7)c$ colliding with each other). The results of Ex. 20 may be useful.

41.* Examples of the limits of Newtonian mechanics

Use the particle speed $\beta = 1/7$ (Ex. 39) as an approximate maximum limit for the validity of Newtonian mechanics. Fill in the table below, following the example of the completed first entry.

Example of motion	β	Is Newtonian analysis of this motion adequate?
Satellite circling the earth at a speed of 18,000 miles per hour.	1/37,200	Yes, because $\beta < 1/7$
Earth circling the sun at an orbital speed of 30 kilometers per second.		
Electron circling a proton in the orbit of smallest radius in a hydrogen atom. (Hint: The speed of the electron in the inner orbit of an atom of atomic number Z, where Z is the number of protons in the nucleus, is derived in Ex. 101 in Chap. 2 (accurate for low velocities) $v = (Z/137)c$ which is accurate for low velocities; for hydrogen $Z = 1$.)		
Electron in the inner orbit of the gold atom, for which $Z = 79$.		
Electron moving with kinetic energy of 5000 electron-volts. (Hint: One electron-volt is equal to 1.6×10^{-19} joules. Try using the Newtonian expression for kinetic energy.)		
A proton or neutron moving with kinetic energy of 10 MeV (million electron-volts) in a nucleus.		

F. SPACETIME PHYSICS: MORE OBSERVATIONS

42. Time dilation with μ-mesons—a worked example

In a given sample of μ-mesons (mu-mesons: elementary particles produced in some nuclear reactions), half will decay to other elementary particles in 1.5 microseconds (measured with respect to a reference frame in which the μ-mesons are at rest). Half of the remainder will decay in the next 1.5 microseconds, and so on.

(a) Consider μ-mesons produced by the collision of cosmic rays with gas nuclei in the atmosphere at a height 60 kilometers above the surface of the earth. The μ-mesons move vertically downward with a speed approaching that of light. Approximately how long will it take them to reach the earth as measured by an observer at rest on the surface of the earth? If there were no time dilation, approximately what fraction of the mesons produced at a height of 60 kilometers would remain undecayed by the time they reached the earth?

(b) Idealize the rather complicated actual experimental situation to following roughly equivalent situation: All the mesons are produced at the same height (60 kilometers); all have the same speed; all travel straight down; of these 1/8 reach sea level without undergoing decay. Question: How can there possibly be so great a discrepancy between the prediction of part a and this observation? And how great is the *difference* between the velocity of these μ-mesons and the velocity of light?†

Solution: The μ-mesons travel with nearly the speed of light. They therefore travel 60 kilometers in approximately

$$\frac{60 \times 10^3 \text{ meters}}{3 \times 10^8 \text{ meters/second}} = 2 \times 10^{-4} \text{ seconds}$$

The "half-life" of μ-mesons is 1.5×10^{-6} seconds as observed in a reference frame in which they are at rest. If there were no time dilation the travel time to earth would be $2 \times 10^{-4}/1.5 \times 10^{-6} = 133$ half-lives. The passage of each half-life reduces the number of remaining μ-mesons by one-half. Therefore after 133 half-lives there should be only the fraction

$$1/2 \times 1/2 \times 1/2 \times 1/2 \cdots = 1/2^{133} \approx 10^{-40}$$

remaining. In fact, there are $1/8 = 1/2^3$ remaining, as determined by experiment (part b). Therefore in the rocket frame in which the μ-mesons are at rest, only 3 half-lives have passed

$$\begin{aligned}\Delta t' &= 3 \times (1.5 \times 10^{-6} \text{ seconds}) \\ &\qquad \times (3 \times 10^8 \text{ meters/second}) \\ &= 1.35 \times 10^3 \text{ meters}\end{aligned}$$

The motion of the meson, seen in the frame of reference attached to it, is naturally zero

$$\Delta x' = 0$$

Therefore the interval of proper time from formation to arrival at the ground is

$$\Delta\tau = [(\Delta t')^2 - (\Delta x')^2]^{1/2} = 1.35 \times 10^3 \text{ meters}$$

But this interval has the same numerical value in the laboratory frame as in the meson frame; thus,

$$\Delta\tau = [(\Delta t)^2 - (\Delta x)^2]^{1/2} = 1.35 \times 10^3 \text{ meters}$$

or

$$[(\Delta x/\beta)^2 - (\Delta x)^2]^{1/2} = 1.35 \times 10^3 \text{ meters} \tag{61}$$

We know the distance of travel in the laboratory frame, $\Delta x = 6 \times 10^4$ meters. Consequently we can find the velocity β from Eq. 61. Square both sides of the equation and divide by $(\Delta x)^2$, finding

$$(1/\beta^2) - 1 = [1.35 \times 10^3/(6 \times 10^4)]^2$$

or

$$\frac{1-\beta^2}{\beta^2} = 5.06 \times 10^{-4}$$

Clearly β is nearly equal to one. Therefore set

$$1 - \beta^2 = (1+\beta)(1-\beta) \approx 2(1-\beta)$$

from which

$$\frac{1-\beta^2}{\beta^2} \approx \frac{2(1-\beta)}{\beta^2} \approx 2(1-\beta) \approx 5 \times 10^{-4}$$

or

$$1 - \beta \approx 2.5 \times 10^{-4}$$

The difference in speed between the mesons and light is given by this small fraction.

†A film about this experiment is available. See David H. Frisch and James H. Smith, "Measurement of the Relativistic Time Dilation Using μ-Mesons," American Journal of Physics, **31**, 342 (May 1963). The original experiment is reported by B. Rossi and D. B. Hall, in Physical Review, **59**, 223 (1941).

43. Time dilation with π^+-mesons

Laboratory experiments on particle decay are much more conveniently done with π-mesons than with μ-mesons, as is seen from the following table.

Particle	Time for half to decay (measured in rest frame)	"Characteristic distance" (speed of light multiplied by foregoing time)
μ-meson (207 times electron mass)	1.5×10^{-6} second	450 meters
π-meson (273 times electron mass)	18×10^{-9} second	5.4 meters

In a given sample of π^+-mesons half will decay to other elementary particles in 18 nanoseconds (measured in a reference frame in which the π^+-mesons are at rest). Half of the remainder will decay in the next 18 nanoseconds, and so on. In the Penn–Princeton proton synchrotron π^+-mesons are produced when a proton beam strikes an aluminum target *inside* the accelerator. Mesons leave this target with nearly the speed of light. If there were no time dilation and if no mesons were removed from the resulting beam by collisions, what would be the greatest distance from the target at which half of the mesons would remain undecayed? The π-mesons of interest in a particular experiment have $\cosh\theta = 1/(1-\beta^2)^{1/2} = 15$ where θ is the velocity parameter. By what factor is the predicted distance from the target for half decay *increased* by time dilation over the previous prediction—that is, by what factor does this dilation effect allow one to increase the separation between his detecting equipment and the target?

44.* Aberration of starlight

The angle between one *remote* star (B) and other *remote* stars (A, C) appears to change from one time of year to another because the earth changes its velocity over a six-month period by 2×30 kilometers per second = 60 kilometers per second. Show that the angle of aberration, ψ (relative to angles as measured by an observer on the sun), is given by the equation $\sin\psi = \beta$. Here β is the speed of the earth in its orbit about the sun. Although the aberration of starlight can be observed experimentally, the aberration angle ψ is so small that it is not at present possible to confirm by experiment that the relativistic prediction above is the correct one—or that the almost equal Newtonian prediction $\tan\psi = \beta$ is not the correct one.

45. Fizeau experiment

Light moves more slowly through a transparent material medium than through a vacuum. Let β' represent this reduced speed of light in the medium. Idealize to a case in which this reduced velocity is independent of the wavelength of the light. Place the medium in a rocket moving at velocity β_r to the right relative to the laboratory frame, and let light travel through the medium, also to the right. Use the law of addition of velocities to find an expression for the velocity β of the light in the laboratory frame. Show that for small relative velocity β_r between the rocket and laboratory frames, the velocity of the light with respect to the laboratory frame is given approximately by the expression

$$\beta \approx \beta' + \beta_r[1 - (\beta')^2] \tag{62}$$

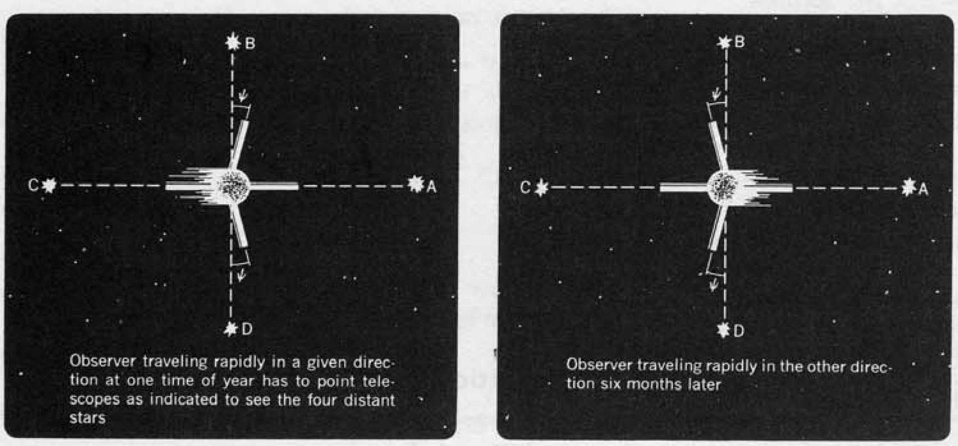


Fig. 61. Aberration of starlight. Both diagrams show positions as observed in a reference frame in which the sun is at rest.

This expression has been tested by Fizeau using water flowing in opposite directions in the two arms of an interferometer similar (but not identical) to the interferometer used by Michelson and Morley (Ex. 33).†

46. Cerenkov radiation

No particle has been observed to travel faster than the *speed of light in a vacuum*. However particles have been observed that travel in a material medium faster than the speed of light *in that medium*. When a charged particle moves through a medium faster than light moves in that medium it radiates *coherent* light in a cone whose axis lies along the path of the particle. (Note the similarity to waves created by a motorboat speeding across calm water!) This is called Cerenkov radiation (C—Russian—is pronounced as "ch"). Let β be the speed of the particle in the medium and β' be the speed of light in the medium. From this information use Fig. 62 to show that the half-angle ϕ of the light cone is given by the expression

(63) $$\cos\phi = \beta'/\beta$$

†H. Fizeau, Comptes Rendus, **33**, 349 (1851). A fascinating discussion (in French) of some central themes in relativity theory—delivered more than fifty years before Einstein's first paper.

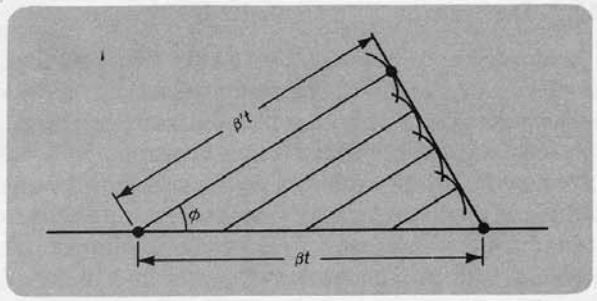


Fig. 62. Calculation of Cerenkov angle ϕ.

Consider the plastic Lucite for which $\beta' = 2/3$. What is the minimum velocity that a charged particle can have if it is to produce Cerenkov radiation in Lucite? What is the *maximum* angle ϕ at which Cerenkov radiation can be produced in Lucite? Measurement of the angle provides a good way to measure the velocity of the particle.†

†For details on the experimental uses of Cerenkov radiation see Chapter VII of *Techniques of High Energy Physics*, edited by David M. Ritson, (Interscience Publishers, New York, 1961).

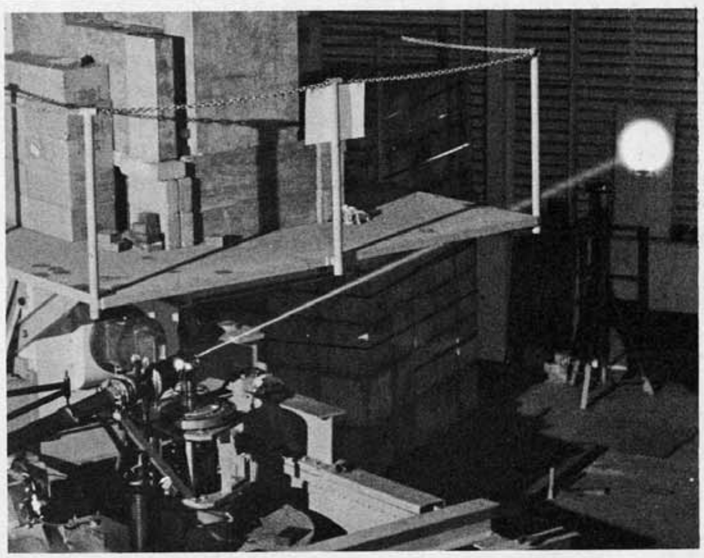


Fig. 63. Cerenkov radiation from a beam of 700-MeV electrons traveling through air. The beam is very much narrower than the circle of Cerenkov light seen on the screen. The beam emerges at the lower left—through a thin aluminum foil—from the vacuum inside the Stanford University linear electron accelerator. The beam itself is visible in the picture through the excitation and ionization it causes as it passes through the air. In addition to this excitation of air molecules the electrons emit Cerenkov radiation in a narrow forward cone. The cone of light from the left hand portion of the beam intercepts the screen in a circular ring which constitutes the outer portion of the disk of light on the screen. Electrons nearer the screen emit radiation at the same angle, which strikes the screen in smaller concentric rings because the emitting electrons are nearer to the screen. The result of radiation from electrons near and far is a solid disk of light. The Cerenkov angle ϕ for the most distant electrons should correspond to the half-angle subtended by the circle at the exit window from the vacuum system. The speed β of the 700-MeV electrons differs from 1—the speed of light—by less than one part in a million (determined using expressions from Chapter 2). Therefore there is little error in giving β the value 1. The speed β' of light in air can be calculated from the observed index of refraction of light in air: $n = 1/\beta' = 1.00029$. The Cerenkov formula becomes

$$\cos\phi = \beta'/\beta \approx \beta' = 1/n = 1/1.00029$$

For small ϕ we can replace these expressions by approximate ones

$$\cos\phi \approx 1 - \phi^2/2 = (1 + 2.9 \times 10^{-4})^{-1} \approx 1 - 2.9 \times 10^{-4}$$

from which the calculated value of the angle is $\phi_{calc} \approx 2.4 \times 10^{-2}$ radian. The distance from exit window to screen is approximately 40 feet and the radius of the spot is about 10.5 inches, or 0.88 foot. The observed angle is thus

$$\phi_{obs} \approx 0.88/40 = 2.2 \times 10^{-2} \text{ radian}$$

which compares well with the calculated value. (The time exposure photograph was taken by A. M. Hudson of Occidental College and is reproduced here with his permission.)

47.* Deflection of starlight by the sun

Estimate the deflection of starlight by the sun using an elementary analysis. Discussion: Consider first a simpler example of a similar phenomenon. An elevator car of width L is released from rest near the surface of the earth. At the instant of release a narrow beam of light is fired horizontally from one wall of the car toward the other wall. After release the elevator car is an inertial frame. Therefore the light beam will cross the car in what is a straight line *with respect to the car*. With respect to the *earth*, however, the beam of light is falling—because the elevator is falling. Therefore, in a gravitational field, a beam of light must fall. As another example a ray of starlight in its passage tangentially across the earth's surface will receive a gravitational deflection (over and above any refraction by the earth's atmosphere). However, the time to cross the earth is so short, and in consequence the deflection so slight, that this effect has not yet been detected on the earth. At the surface of the sun, however, the acceleration of gravity has the much greater value of 275 meters per second per second. Moreover, the time of passage across the surface is much increased because the sun has a greater diameter, 1.4×10^9 meters. Determine an "effective time of fall" from this diameter and the speed of light. From this time of fall deduce the net *velocity* of fall toward the sun produced by the end of the whole period of gravitational interaction. (The maximum acceleration acting for this "effective time" produces the same net effect [calculus proof!] produced by the actual acceleration—changing in magnitude and direction along the path—in the entire passage of the ray through the sun's field of force.) Comparing this lateral velocity with the forward velocity of the light deduce the *angle of deflection*. The accurate analysis of special relativity gives the *same result*. However, Einstein's 1915 general relativity predicted a previously neglected effect, associated with the change of *lengths* in a gravitational field that produces something like a supplementary *refraction* of the ray of light and *doubles* the predicted deflection. (Deflection observed in 1947 eclipse of the sun: $(9.8 \pm 1.3) \times 10^{-6}$ radian; in the 1952 eclipse: $(8.2 \pm 0.5) \times 10^{-6}$ radian.)

G. GEOMETRIC INTERPRETATION

48. Geometric interpretation

Develop a geometric interpretation of the Lorentz transformation using the following outline.

(a) Show that in the laboratory spacetime diagram the world line of the origin of the rocket frame will be the line marked t' in Fig. 64. This is the locus of all events that occur at the origin of the rocket frame, that is, *it is the rocket t' axis*. Show that the locus of events that occur at $x' = 1$ meter in the rocket frame is a line that parallels the t' axis in Fig. 64, and similarly for $x' = 2, 3, 4$ meters.

(b) Show that the slope of the t' axis relative to the t axis in Fig. 64 is given by the expression (meters of distance traveled for each)/(meter of light-travel time) $= \beta_r = \tanh \theta_r$. What happens to the slope β_r in the two cases: (1) the rocket travels very slowly and (2) the rocket travels at a speed very close to the speed of light.

(c) Now for the crucial step! Where shall we locate the rocket x' axis in the laboratory spacetime diagram? The principle of relativity says that the measured speed of light must be the same in the two frames. The dotted line in Fig. 65 is the world line of a flash of light. Show that the principle of relativity requires that the rocket x' axis be tilted upward at the same slope as

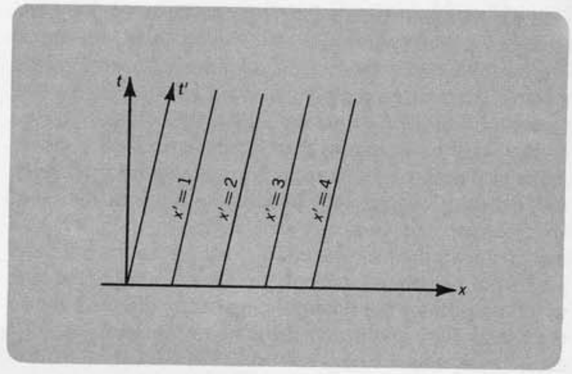
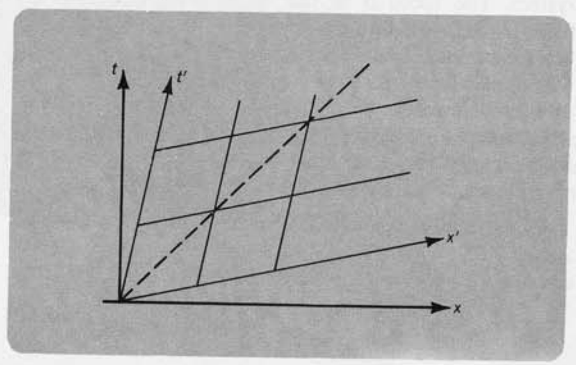


Fig. 64. Location of the rocket time axis in the laboratory spacetime diagram.

Fig. 65. Location of the rocket space axis in the laboratory spacetime diagram.



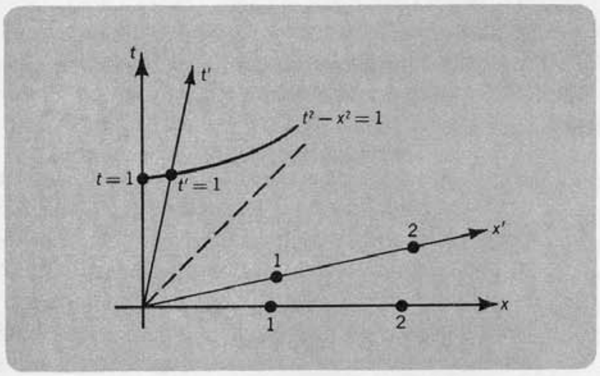


Fig. 66. Calibration of rocket space and time axes.

the rocket t' axis is tilted to the right. Show that the loci of events that occur at rocket times $t' = 1, 2, 3$ meters respectively lie parallel to the rocket x' axis as shown.

(d) Calibrate the rocket axes! Draw the hyperbola $t^2 - x^2 = 1$ (Fig. 66). At the place where the hyperbola crosses the laboratory t axis (where $x = 0$), we have $t = 1$ meter of time. But the interval $t^2 - x^2$ is an invariant so that $(t')^2 - (x')^2 = 1$ also. Therefore at the place where the hyperbola crosses the rocket t' axis (where $x' = 0$), we have $t' = 1$ meter of time. Because of the symmetry and the linearity of the transformation equations, we can use the distance along the rocket t' axis from the origin to $t' = 1$ as a unit distance to lay off along *both* the t' and the x' axes. This completes the derivation of the construction. Next: apply it!

(e) Show that if two events are simultaneous in the laboratory frame they will lie on a line parallel to the laboratory x axis of the spacetime diagram (Fig. 67). Show that if two events are simultaneous in the rocket frame they will lie on a line parallel to the rocket x' axis of the spacetime diagram. Hence the two observers will not necessarily agree on which events are simultaneous. This is the *relative synchronization of clocks*.

(f) Using lines of simultaneity in Fig. 67, show that at rocket time $t' = 1$ meter, the observer in the rocket frame determines that the clock at the laboratory origin has not yet reached one meter of time (i.e., the laboratory clock runs slow), whereas the observer in the laboratory frame observes that the clock at the laboratory origin already reads *more* than one meter of time (i.e., the rocket clock runs slow). This is *time dilation.*

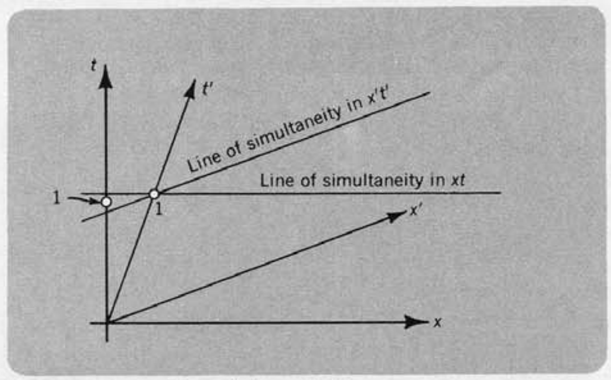


Fig. 67. Illustration of time dilation.

(g) A meter stick lies at rest in the laboratory frame with one end at the origin of that frame (Fig. 68). Measurement of its length in the laboratory frame will give a result like *ab* in Fig. 68. Measurement of its length in the *rocket frame* (i.e., determining the position of the endpoints at the "same time") will give a result like *de* in the figure. Show that this measurement results in an observed *Lorentz contraction* in the rocket frame. Using Fig. 69 show that a meter stick at rest in the rocket frame with one end at the origin of that frame will be Lorentz contracted when observed in the laboratory frame.

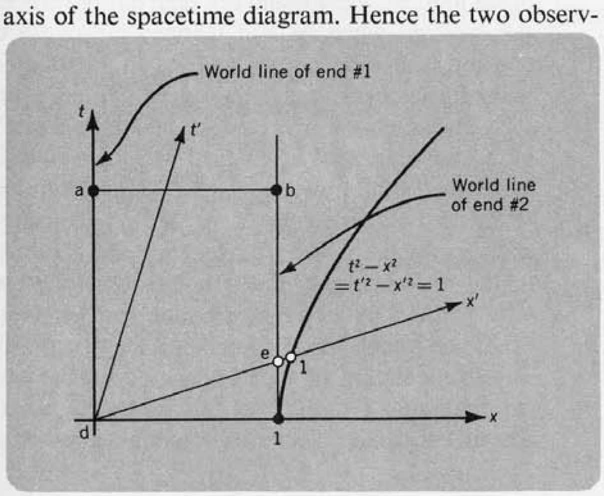


Fig. 68. A meter stick at rest in laboratory frame appears Lorentz-contracted when observed in rocket frame.

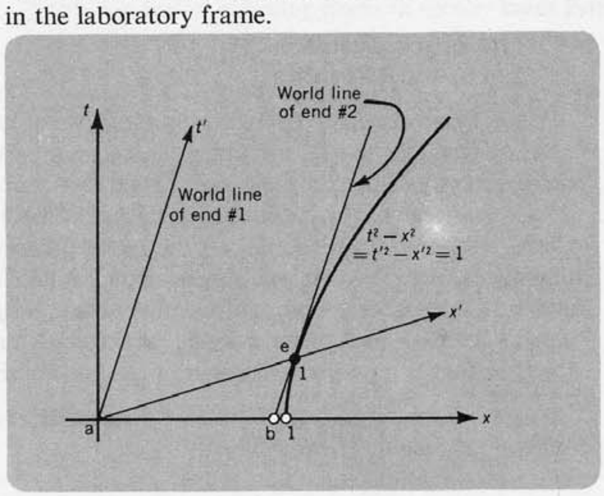


Fig. 69. A meter stick at rest in rocket frame appears Lorentz-contracted when observed in laboratory frame.

(h) Sketch spacetime diagrams for the relativity of simultaneity, time dilation, and Lorentz contraction in the limiting cases that the relative velocity between laboratory and rocket frames is very small and very large.

(i) Return to the spacetime diagram of Fig. 22 in the chapter, which describes the motion of particles and light flashes in two dimensions. Show that the rocket "plane of simultaneity" is tilted relative to the laboratory plane of simultaneity. Explain the implications of this tilt for the relative simultaneity of events that occur at different positions on the x axis of the laboratory spacetime diagram, and for the relative simultaneity of events that occur at different positions on the y axis of the laboratory spacetime diagram.

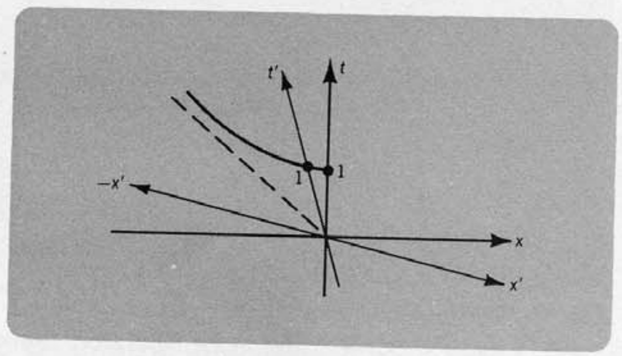


Fig. 70. Location of space and time axes for rocket frame moving in negative laboratory x direction.

(j) Consider a rocket frame moving in the *negative* x direction in the laboratory frame. Verify the features of Fig. 70, in particular the *opposite* sense of the relative synchronization of clocks and the *same* sense of time dilation when compared with the rocket moving in the positive x direction.

49. The clock paradox II—a worked example†

When Peter returned from his fourteen years of traveling (Ex. 27) he was still young enough to learn some relativity. But the more he studied the more puzzled he became. He and his brother Paul, being in relative motion, "each should see the other's clocks running slow." This simple slogan, put in Paul's mouth, made it easy enough to understand why "Peter's clocks—and Peter's aging processes—ran slow," so that Peter was the younger of the two on his return. "But if the slogan is valid," Peter asked, "then would not *I*—if I had investigated—have found *Paul's* clocks running slow? So how did he age more than I?" *Question:* What is the way out of Peter's difficulties?

Solution: As Peter studied more, with this paradox worrying him, he learned that words like "observer" and "observed time" do not have the simple meaning he had at first attributed to them. He should not think of how he might directly have kept day-to-day track of Paul's aging back on earth, either by radio messages or by other methods. That procedure, while conceivable, does not lend itself to the simplest analysis, Peter discovered. The observer in relativity theory, he found, is to be understood as a whole framework of rods and recording clocks moving along with uniform velocity—with the same velocity as Peter himself as he recedes from the earth, $\beta_r = 24/25 = 0.96$. That parade of clocks ("Peter's clocks and Peter's reference frame") zooms by the earth. As each clock passes Paul it punches out (1) the reading of Paul's clock and (2) its own reading and location. The shorthand phrase "Peter observes Paul" means that Peter collects these cards—or the information on them—at some later time.

"So what?" Peter asked himself at this point. "In any case I know that the reading of Paul's clock increases from one punchout to the next only $(1 - \beta_r^2)^{1/2} = 7/25$ as much as the increase in readings of my own clock. So Paul is the man who should have been younger at the end of my journey, not me. But look at his gray hair! Where am I going wrong?"

Running over in his mind once again the events of his journey, Peter could not help but remember the moment when he had stopped his outward trip and started his return to the earth. "*I* stopped and *I* turned back; but," he suddenly asked himself, "what about my inertial reference frame? How can an inertial frame turn back? He looked into this issue more and more carefully. He found himself forced to conclude that the reference frame employed for the first part of his flight—and especially the lattice clock alongside him that had recorded information for the seven outbound years—must have kept on their swift way like a stream of superhighway traffic as one car makes a U-turn into the returning lanes. Another stream of clocks accompanied him home

†See E. Lowry, American Journal of Physics, **31**, 59 (1963).

—a second inertial reference frame. For all the seven years of return one of these clocks remained faithfully alongside. When it took over escort duty, it adopted the seven-year reading of the outbound clock. It read fourteen years at the time when Peter rejoined Paul.

The inbound parade of clocks was passing the earth all these seven years. One by one as they went by they punched out their readings and the readings of Paul's clock. The punch cards made a growing pile on the ground at Paul's feet. As those seven years went by for Peter's inbound escort, the cards showed that Paul's clocks ran off only 7/25 of this time; that is (7/25) of 7 years or 1.96 years.

"What on earth is the matter with my reasoning?" Peter asked aloud at this point. "Now I find myself concluding that Paul should have aged 1.96 years on my outbound trip, and 1.96 years on my inbound trip, or altogether 3.92 years. Yet I *know* I aged fourteen years, and I *know* he aged more than I did. What have I overlooked?" So saying, he drew a spacetime diagram (Fig. 71), and at least had the answer to his difficulty—the time AB that he had so far left out of account. This time, Peter saw, corrects for the difference between the standards of simultaneity of his outgoing and returning reference frames. A separate calculation, using the results of Ex. 11, gives for this time the value 46.08 years. This supplement has to be added to Paul's aging as measured by Peter's two sets of recording clocks. Peter's final calculation for Paul's age (including his age of 21 years when the trip began) gave

$$21 + 1.96 + 46.08 + 1.96 = 71 \text{ years}$$

He could thankfully rejoice in his own comparative youth of $21 + 14 = 35$ years (uncorrected for the time required to learn spacetime physics!). The present analysis does not purport to be the simplest way to calculate the aging of the twins. For that one goes back to Paul's analysis, outlined in Ex. 27. There one has to consider only a single inertial reference frame, the frame with its origin at Paul. The present analysis illustrates how *any* correct method of analysis leads to the same result as any other correct method of analysis.

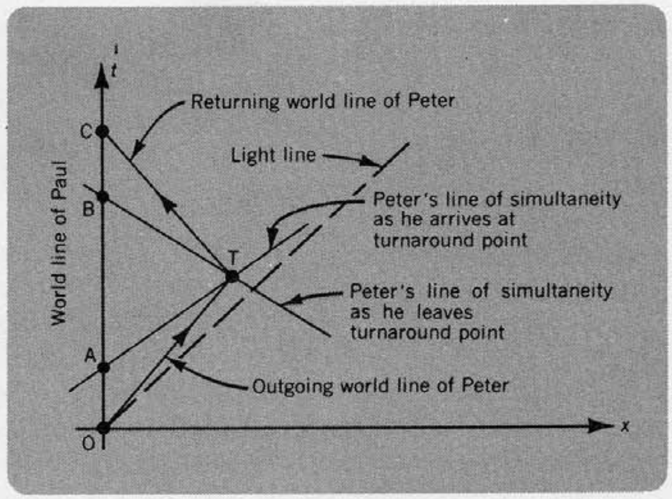


Fig. 71. Peter's bookkeeping on Paul's aging process. During Peter's outbound journey (OT in diagram) his clock flashes a new year seven times. An array of synchronized clocks escort him. Each makes its own seventh year flash somewhere along the "line of simultaneity" AT and punches out a record. The Peter clock which punches out a record at A sees Paul's clock reading only 1.96 years ("slowing of a clock as viewed from a moving reference frame"). On the return journey a different array of synchronized clocks escorts Peter ("second inertial reference frame"). Each of them flashes a seven year sign as it crosses the line of simultaneity BT. The one which travels alongside Peter makes seven more flashes along the world line TC, the last of them signaling fourteen years of travel just as Peter rejoins Paul at C. During the period BC, while the clocks of Peter's inbound reference frame indicate the passage of seven years, Paul has aged only another 1.96 years (again the "slowing of a clock as viewed from a moving reference frame"). But the bookkeeping done so far by Peter's two inertial reference frames is incomplete. Neither one of them does the job of counting the time lapse AB. It is 46.08 years ("correction for change in standard of simultaneity" between Peter's outgoing and incoming inertial reference frames). Thus the slowing of Paul's clocks as observed by Peter's two sets of recording clocks in no way keeps Peter from ending up younger than Paul.

H. FREE-FOR-ALL!

50. Contraction or rotation?†

Consider a cube, at rest in the rocket frame, whose edge measures 1 meter in that frame. In the *laboratory* frame the cube is Lorentz contracted in the direction of motion, as shown in Fig. 72. This Lorentz contraction can be determined, for example, from the locations of four clocks at rest and synchronized in the laboratory frame with which the four corners of the cube, E, F, G, H, coincide *when all four clocks read the same time*. In this way time lags in the travel of light from different corners of the cube are eliminated from the observation procedure. Now for a different observing procedure!

Stand in the laboratory frame and *look* at the cube with one eye as the cube passes overhead (Fig. 72). What one sees at any time is light *that enters his eye at that time, even if it left the different corners of the cube at different times*. Hence, what one *sees* visually may not be the same as what he *observes* using a latticework of clocks. If the cube is viewed from the bottom then the distance GO is equal to the distance HO, so light that leaves G and H simultaneously will arrive at O simultaneously. Hence, when one sees the cube to be overhead he will see the Lorentz contraction of the bottom edge.

(a) Light from E that arrives at O simultaneously with light from G will have to leave E earlier than light from G left G. How much earlier? How far has the cube moved in this time? What is the value of the distance x in Fig. 73?

(b) Suppose that one chooses to interpret the projection in Fig. 73 as a rotation of a cube that is *not* Lorentz contracted. Find an expression for the angle of apparent rotation ϕ of this uncontracted cube in Fig. 74. Interpret this expression for the two limiting cases $\beta \longrightarrow 0$ and $\beta \longrightarrow 1$.

(c) Is the word "really" an *appropriate* word in the following quotations?

(1) An observer in the rocket frame says, "The cube is *really* neither rotated nor contracted."

(2) An observer using the laboratory latticework of clocks says, "The cube is *really* Lorentz con-

†For a more complete treatment of this topic, and references, see Edwin F. Taylor, *Introductory Mechanics*, (John Wiley and Sons, New York, 1963), p. 346.

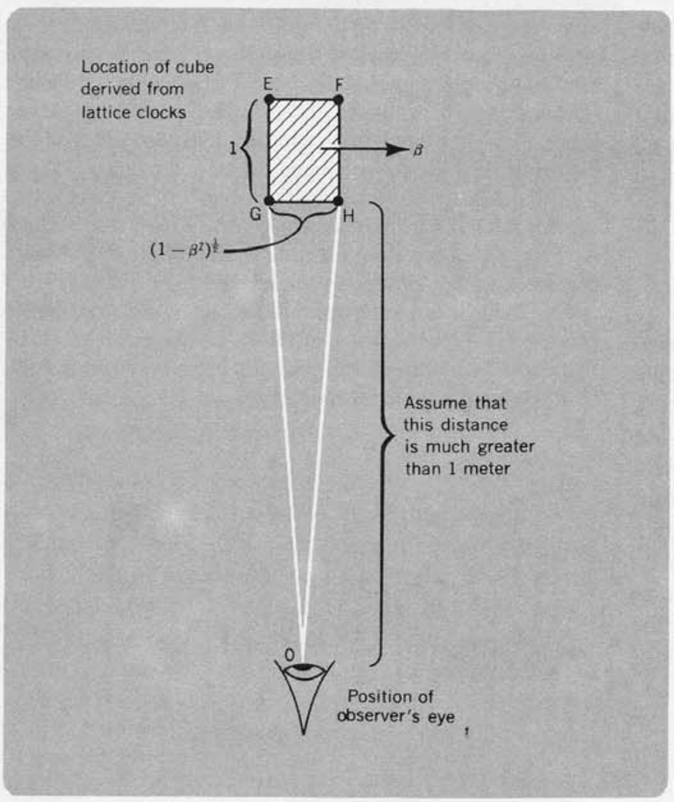


Fig. 72. Position of eye of visual observer watching "cube" pass overhead.

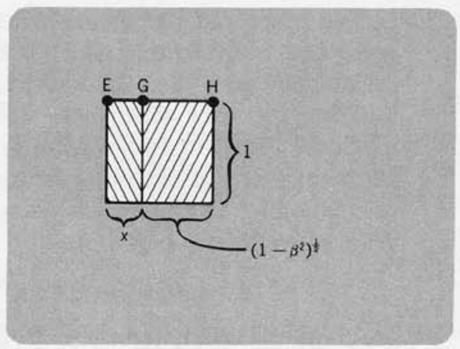


Fig. 73. What visual observer sees as he looks up from below.

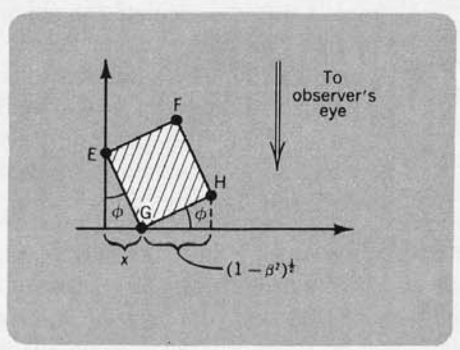


Fig. 74. How visual observer can interpret the projection of Fig. 73.

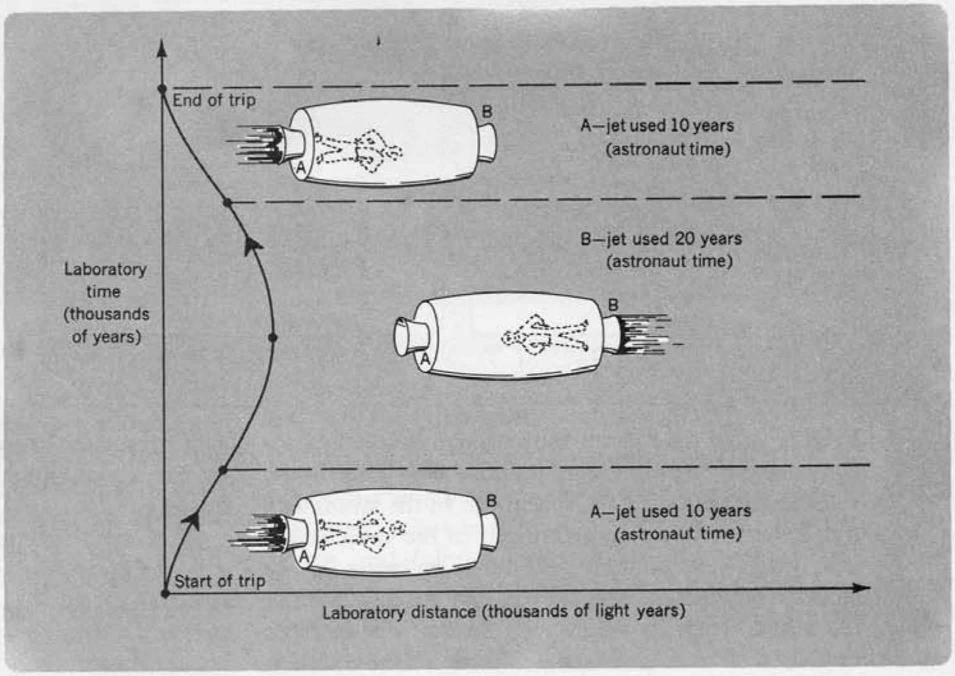


Fig. 75. Round-trip world line of rocket which experiences constant acceleration or deceleration.

tracted but not rotated."

(3) The visual observer in the laboratory frame says, "The cube is *really* rotated but not Lorentz contracted."

What can one rightfully say—in a sentence or two—to make each observer think it reasonable that the other observers should come to conclusions different from his own?

51.** Clock paradox III

Can one go to a point 7000 light years away—*and return*—without aging more than 40 years? "Yes" is the conclusion reached by an engineer on the staff of a large aviation firm in a recent report. In his analysis the traveler experiences a constant "1-g" acceleration (or deceleration, depending upon the stage reached in his journey—see spacetime diagram of Fig. 75). Assuming this limitation, is he right in his conclusion? (For simplicity, limit attention to the first or "A"-jet phase of the motion—the first 10 years of astronaut time—and double the distance covered in that time to find how far it is to the most remote point reached in the course of the journey.)

(a) The acceleration is *not* $g = 9.8$ meters per second per second relative to the laboratory frame. If it were, how many times faster than light would the spaceship be moving at the end of ten years (1 year $= 31.6 \times 10^6$ seconds)? *If the acceleration is not specified with respect to the laboratory, then with respect to what is it specified?* Discussion: Look at the bathroom scales on which one is standing! The rocket jet is always turned up to the point where these scales read one's *correct* weight. Under these conditions one is being accelerated at $g = 9.8$ meters per second per second with respect to a spaceship that (1) instantaneously happens to be riding alongside with identical velocity, but (2) is *not* being accelerated, and, therefore (3) *provides the* (momentary) *inertial frame of reference relative to which the acceleration is g*. (Hereafter this acceleration is translated from g—expressed in meters per second per second—to $g^* = g/c^2$—measured in meters of distance per meter of time per meter of time.)

(b) *How much velocity does the spaceship have after a given time?* This is the moment to object to the question and to rephrase it. *Velocity* β is not the simple quantity to analyze. The simple quantity is the *velocity parameter* θ. It is simple because it is *additive* in this sense: Let the velocity parameter of the spaceship in Figure 76 with respect to the imaginary instantaneously comoving inertial frame change from 0 to $d\theta$ in an astronaut time $d\tau$. Then the velocity parameter of the spaceship with respect to the *laboratory* frame changes in the same astronaut time from the initial value θ to the subsequent value $\theta + d\theta$. Now relate $d\theta$ to the acceleration g^* in the instantaneously comoving inertial frame. In this frame $g^* d\tau = d\beta = \tanh(d\theta) \approx d\theta$ so that

$$d\theta = g^*\, d\tau \qquad (64)$$

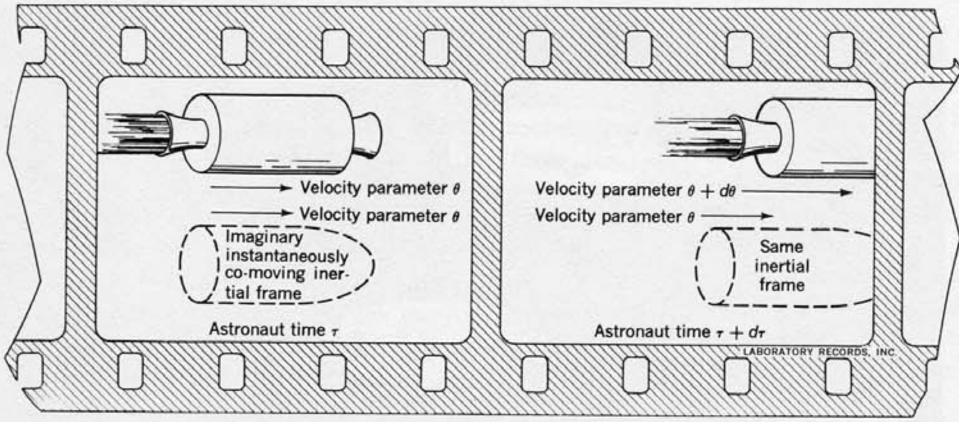


Fig. 76. Laboratory record of accelerating rocket.

Each lapse of time $d\tau$ on the astronaut's watch is accompanied by an additional increase $d\theta = g^* \, d\tau$ in the velocity parameter of the spaceship. In the laboratory frame the total velocity parameter of the spaceship is simply the sum of these additional increases in the velocity parameter. Assume that the spaceship starts from rest. Then its velocity parameter will increase linearly with *astronaut* time according to the equation

$$\theta = g^* \tau \tag{65}$$

This expression gives the velocity parameter θ of the spaceship in the *laboratory* frame at any time τ in the *astronaut's* frame.

(c) *What laboratory distance x does the spaceship cover in a given astronaut time* τ? At any instant the velocity of the spaceship in the laboratory frame is related to its velocity parameter by the equation $dx/dt = \tanh \theta$ so that the distance dx covered in *laboratory* time dt is

$$dx = \tanh \theta \, dt$$

Remember that the time between ticks of the astronaut's watch $d\tau$ appear to have the larger value dt in the laboratory frame (time dilation) given by the expression

$$dt = \cosh \theta \, d\tau$$

Hence the laboratory distance dx covered in *astronaut* time $d\tau$ is

$$dx = \tanh \theta \cosh \theta \, d\tau = \sinh \theta \, d\tau$$

Use the expression $\theta = g^* \tau$ from part b

$$dx = \sinh (g^* \tau) \, d\tau$$

Sum (integrate) all these small displacements dx from zero astronaut time to a final astronaut time to find

$$x = \frac{1}{g^*} [\cosh (g^* \tau) - 1] \tag{66}$$

This expression gives the *laboratory distance x* covered by the spaceship at any time τ in the astronaut's frame.

(d) Convert g^* (in meters per meter per meter) to $g = g^* c^2$ (meters per second per second) and τ (meters) to $\tau_{sec} = \tau / c$ (seconds) in the expression of part c. Determine whether the engineer is correct in his conclusion reported at the beginning of this exercise. (One year is 31.6×10^6 seconds).

52.* The tilted meter stick

A meter stick that lies parallel to the x axis moves in the y direction of the laboratory frame with speed β^y. In the rocket frame the stick is tilted upward in the positive x' direction. Explain why this is, first without using any equations. Let the center of the meter stick pass the point $x = y = x' = y' = 0$ at a time $t = t' = 0$, as shown in the figures. Next calculate the angle θ' at which the meter stick is inclined to the x' axis in the rocket frame. Discussion: Where and when does the right end of the meter stick cross the x axis as observed in the *laboratory* frame? Where and when does the right end of the meter stick make this crossing as observed in the *rocket* frame? The experimentally observed Thomas precession of the electron in an atom—described in Ex. 103—can be explained in the same way as the phenomenon of the tilted meter stick.

53.* The meter-stick paradox†

Note: Ex. 52 should be completed before Ex. 53.

A meter stick lies along the x axis of the laboratory frame and approaches the origin with velocity β_r. A very thin plate parallel to the xz laboratory plane moves upward in the y direction with speed β^y. The

†See R. Shaw, American Journal of Physics, **30**, 72 (1962).

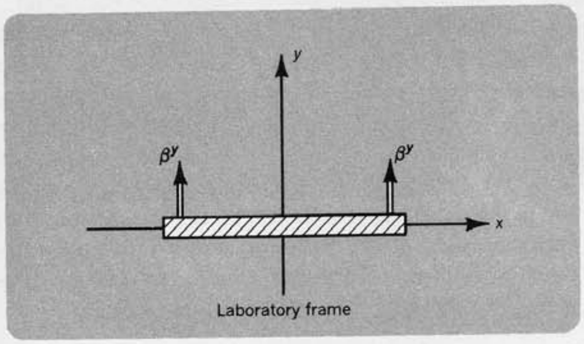


Fig. 77, A. Meter stick moving transverse to its length as observed in laboratory frame.

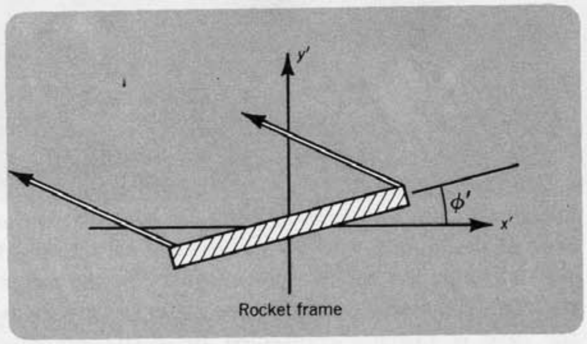


Fig. 77, B. Meter stick as observed in rocket frame.

plate has a circular hole with a diameter of one meter centered on the y axis. The center of the meter stick arrives at the laboratory origin at the same time in the laboratory frame as the rising plate arrives at the plane $y = 0$. Since the meter stick is Lorentz contracted in the laboratory frame it will easily pass through the hole in the rising plate. Therefore there will be no collision between meter stick and plate as each continues its motion. However, someone who objects to this conclusion can make the following argument: In the *rocket* frame in which the meter stick is at rest the meter stick is not contracted, while in this frame the hole in the plate *is* Lorentz contracted. Hence the full-length meter stick cannot possibly pass through the contracted hole in the plate. Therefore *there must be a collision* between the meter stick and the plate. Resolve this paradox using your answer to the preceding problem. Answer unequivocally the question: Will there be a collision between the meter stick and the plate?

54.** The thin man and the grid†

A certain man walks very fast—so fast that the relativistic length contraction makes him very thin. In the street he has to pass over a grid. A man standing at the grid fully expects the fast thin man to fall through the holes in the grid. Yet to the fast man he himself has his usual size and it is the *grid* that has the relativistic contraction. To him the holes in the grid are much narrower than to the stationary man, and he certainly does not expect to fall through them. Which man is correct? The answer hinges on the relativity of rigidity.

Idealize the problem as a one-meter rod sliding lengthwise over a flat table. In its path is a hole one meter wide. If the Lorentz contraction factor is ten, then in the table (laboratory) frame the rod is 10 centimeters long and will easily drop into the one-

†W. Rindler, American Journal of Physics, **29**, 365 (1961).

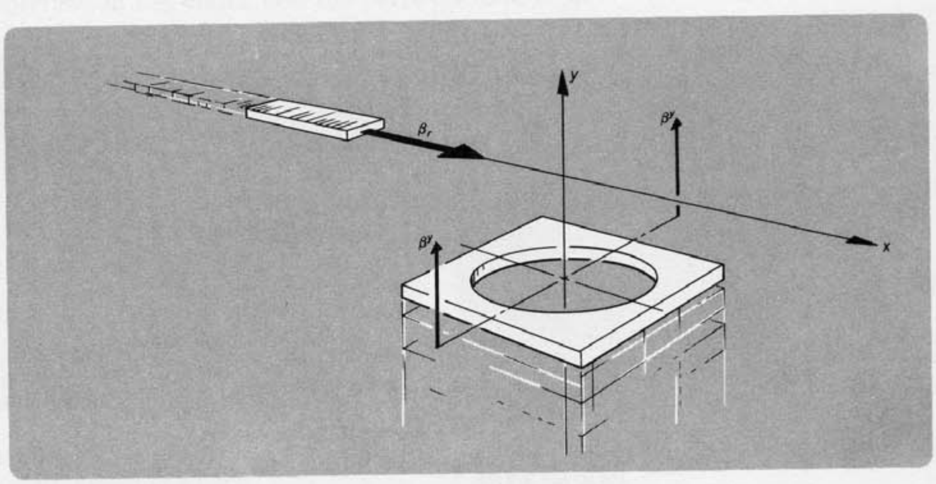


Fig. 78. Will the "meter stick" pass through the "one-meter-diameter hole" without collision?

meter hole. Assume that in the laboratory frame the meter stick moves fast enough so that it remains essentially horizontal as it descends into the hole (no "tipping" in the laboratory frame). Write an equation in the laboratory frame for the motion of the bottom edge of the meter stick assuming that $t = t' = 0$ at the instant that the *back* end of the meter stick leaves the edge of the hole. For small vertical velocities the rod will fall with the usual acceleration g. In the meter stick (rocket) frame the rod is one meter long whereas the hole is Lorentz contracted to a 10-centimeter width so that the rod cannot possibly fit into the hole. Transform the laboratory equations into the rocket frame and show that the rod will "droop" over the edge of the hole in that frame—that is, it will not be rigid. Will the rod ultimately descend into the hole in both frames? Is the rod *really* rigid or nonrigid during the experiment? Is it possible to derive any physical characteristics of the rod (e.g. its flexibility or compressibility) from the description of its motion provided by relativity?