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Spacetime Physics

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SOLUTIONS: EXERCISES FOR CHAPTER 1

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ANSWERS TO THE EXERCISES OF CHAPTER 1

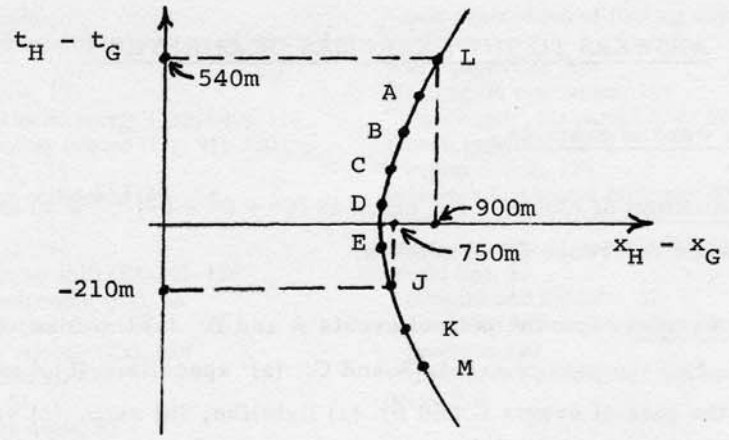
1. Space and time--a worked example.

2. Practical synchronization of clocks. Set clock at $(6^2 + 8^2 + 0^2)^{1/2} = 10$ meters. Press "go" button on clock when reference flash arrives.

3. Relations between events. For the pair of events A and B: (a) timelike, (b) 4 meters of proper time, (c) yes. For the pair of events A and C: (a) spacelike, (b) 4 meters of proper distance (c) no. For the pair of events C and B: (a) lightlike, (b) zero, (c) yes, they can be connected only by a single light ray.

4. Simultaneity. "Simultaneously" is a word appropriate for describing the relationship between (A strikes B) and (C strikes D) only in a particular inertial reference frame. In order to make a statement about the relationship between these two events independent of any choice of frame of reference, one says "(A strikes B) and (C strikes D) are separated by a spacelike interval of one hundred million miles."

5. Temporal order of events. Case of lightlike separation: If a light ray can travel directly from G to H then this light ray can travel directly from G to H! This is a statement about physics that has nothing to do with any choice of inertial reference frame. But then H is later than G in every inertial frame, as was to be shown. Case of timelike separation: H lies within the forward light cone of G in one inertial frame. Therefore it is possible for a particle to move directly from G to H at a uniform velocity less than the speed of light, as recorded in that frame. But the fact that the particle can move directly from G to H has nothing to do with any choice of inertial frame. Therefore H is later than G in every inertial frame, as was to be shown. Case of spacelike separation: The separation between two events must be lightlike or timelike or spacelike; there are no other possibilities. Therefore we need to show that two events with a spacelike separation do not have a unique temporal order if we are to show that this temporal order is unique only for events with lightlike and timelike separations. As an example consider the laboratory coordinate separations $x_H - x_G = 900$ meters, $t_H - t_G = 540$ meters. The spacelike interval is $[(900 \text{ meters})^2 - (540 \text{ meters})^2]^{1/2} = 720$ meters. The same events seen from a frame moving rapidly to the right have a lesser time separation, but the interval remains unchanged. In whatever frame the coordinates are measured, the components of the separation lie on the hyperbola $(x_H - x_G)^2 - (t_H - t_G)^2 = (720 \text{ meters})^2$. (See diagram.) When the speed of the frame is great enough relative to the laboratory frame (example: frame J), event H is seen to occur before event G. A similar analysis holds--and a hyperbola similar to that in the diagram can be drawn--for any two events separated by a spacelike interval, In brief, when G and H are separated by a spacelike interval, G can be made to appear arbitrarily early before H, or arbitrarily



Space and time coordinates separating two events as affected by choice of frame of reference: L, laboratory frame; A, frame moving "slowly" to right relative to laboratory frame; B, C, D, ... , frames moving at higher and higher speeds to right relative to laboratory frame; J, a frame in which the coordinates happen again to be round numbers.

late after H, according as the speed of the observing frame is made sufficiently great either to the right or to the left relative to the laboratory frame.

6. The expanding universe. (a) From the middle section of Figure 35 the proper time between flashes is given by the expression

$$\Delta\tau = \left[(\Delta t)^2 - (\Delta x)^2\right]^{1/2} = \left[(\Delta t)^2 - (\beta\Delta t)^2\right]^{1/2} = \Delta t(1 - \beta^2)^{1/2}$$

From the right hand section of Figure 35 the time lapse between reception of two sequential flashes is given by the expression

$$\Delta t_{reception} = \Delta t + \beta\Delta t = \Delta t(1 + \beta)$$

Eliminate Δt between these two equations and solve to obtain the recession velocity, β:

$$\beta = \frac{(\Delta t_{reception})^2 - (\Delta\tau)^2}{(\Delta t_{reception})^2 + (\Delta\tau)^2}$$

The distance from one's own fragment to the bomb fragment at which one is looking will be given by the time elapsed since the explosion multiplied by the speed of recession of that fragments from one's own fragment.

(b) Find recession velocity of star from foregoing formula. Put $\Delta\tau$ equal to the proper period of the light and $\Delta t_{reception}$ equal to the observed period of light from the distant source. If the universe exploded from a negligibly small initial volume, it follows that now, a time T later, the present distance of each star (or galaxy) will be βT: twice as great for a galaxy receeding at twice the speed. The distance of the galaxy at the earlier time of emission of the light seen here now was $\beta T/(1 + \beta)$. The red shift factor $\Delta t_{reception}/\Delta\tau$ exceeds 3 for the most rapidly receding sources now known (the so-called quasistellar sources), but their distances are not known. Independent distance determinations are limited at present to sources receding only at $\beta = 0.2$ and less. From these distances plus the observed red shifts one finds T to be about 10 to 14×10^9 years.

7. Proper time in communication. First question: true. Second question: No, the proper time is positive. One way to see this is to notice that the reflection back and forth between mirrors allows time for a particle emitted with the flash from the sun to arrive at the point of absorption at the same time as the light flash. The proper time between the emission and arrival of the particle must be greater than zero. Third question: No, the proper time is greater than zero.

8. Data-collection and decision-making. The lag time is R meters of light-travel time provided that direct light flashes are used for communication. All other methods of communication will result in a greater lag time. Observers will have 3.4 seconds to take evasive action, 0.4 seconds longer than the required 3 seconds.

9. Lorentz contraction--a worked example.

10. Time dilation. (a) One may choose, for example, the events of breaking notches in paper masks shown in Figure 38. (b) By definition $\Delta x' = 0$. Substituting this value into Eq. 42, one obtains Eq. 44. (c) The principle of relativity is not violated, because there is symmetry between frames; a single clock at rest in the laboratory frame is observed to run slow when compared at coincidence with a series of clocks at rest in the rocket frame (part d). The discussion of part d in the previous worked exercise may be useful. (d) By definition $\Delta x = 0$. Substitute this value into Eq. 39 to obtain Eq. 45.

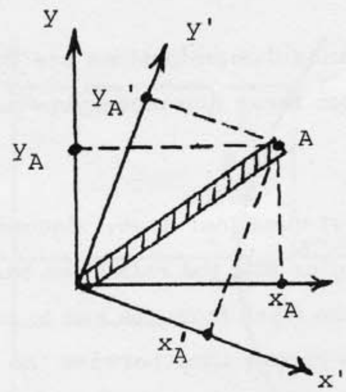
11. Relative synchronization of clocks. (a), (b), and (c) If $\Delta x = 0$ and $\Delta t = 0$, then the Lorentz transformation equations (39) imply $\Delta t' = 0$ for all rocket frames. This is true if Δy and Δz are both equal to zero--and also if these quantities are not equal to zero (part b). If $\Delta t = 0$ but $\Delta x \neq 0$, then

$$\Delta t' = -\Delta x \sinh \theta_r \neq 0$$

Equation 46 is obtained by applying corresponding conditions ($t = 0$) to Eq. 37. (d) Apply the condition $t' = 0$ Eqs. 36 to obtain Eq. 47. (e) A choice of the positive rocket x'-direction in the direction of motion of the laboratory frame results in a reversal of sign in Eq. 47, making this equation symmetrical with Eq. 46. (f) In order to make measurements at several different places in the rocket frame at $t' = 0$ (simultaneously in that frame), one needs to use several recording clocks. A better statement: "Let recording rocket clocks be arranged to be near every laboratory clock at the origin of rocket time ($t' = 0$). Let them photograph the faces of the laboratory clocks at this time. Then the laboratory clock readings so recorded will not all be $t = 0$."

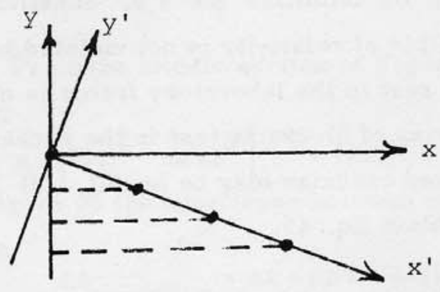
12. Euclidean analogies

(a) and (b)



The analogy consists of comparing the x-coordinates in the Euclidean plot with the x-coordinates in the Lorentz spacetime diagram; comparing the y Euclidean coordinate with the t coordinate in the spacetime diagram. Thus in the diagram above, the distance x_A' is less than the distance x_A, corresponding to the difference in length of a moving rod as observed in rocket as opposed to laboratory frame. Similarly the dilation of time has its analogy to the difference in the y coordinates y_A' and y_A in the two Euclidean coordinate systems. The Euclidean invariant is the length of the rod as calculated from the coordinates of the endpoints in any coordinate system. The Lorentz invariant is the interval between two events as calculated from observations in any inertial reference frame.

(c)



Points with the coordinate $y' = 0$ do not all have the coordinate $y = 0$. In the same way events occuring at $t' = 0$ do not all have the coordinate $t = 0$.

13. Lorentz contraction II. Concentrate on these two events: the two ends of the meter stick pass the origin of laboratory space coordinates. In the rocket frame these two events are separated by minus one meter of distance (minus because the laboratory moves in the negative x' direction in the rocket frame), and by (1 meter)/(relative speed) of time.

$$\Delta x' = -1 \text{ meter}$$
$$\Delta t' = (1 \text{ meter})/\beta_r$$

In the laboratory frame the two events occur at the same place separated by a time Δt which we are instructed by the problem to set equal to L/(relative speed) where L is the "length" of the meter stick as measured (by this means) in the laboratory frame. Substitute these values into the Lorentz transformation equation (16) expressed in terms of the relative speed

$$\Delta t = L/\beta_r = \frac{\beta_r(-1 \text{ meter}) + (1 \text{ meter})/\beta_r}{(1-\beta_r^2)^{1/2}}$$

from which

$$L = (1-\beta_r^2)^{1/2} \text{ meters}$$

which is the "Lorentz-contracted length" as observed in the laboratory frame (Eq. 38).

14. Time dilation II. According to the statement of the problem, $\Delta x' = 0$ while $\Delta t' \neq 0$. The distance between the two events in the laboratory frame can be found from the Lorentz transformation equation

$$\Delta x = 0 + \Delta t' \sinh\theta_r$$

We are told to "measure" the laboratory time between the two events by dividing this laboratory distance by the relative speed between frames

$$\Delta t = \Delta x/\beta_r = \Delta x/\tanh\theta_r = \Delta t' \cosh\theta_r$$

which is the expression for time dilation (Eq. 44).

15. Lorentz transformation equations with time in seconds. Simply write ~~$t = t_{sec}/c$~~ $t = c\,t_{SEC}$ and $\beta_r = v_r/c$ in Equations 37. The inverse equations (see Eqs. 36 and 16) are

$$x = x' \cosh\theta_r + ct_{sec}' \sinh\theta_r = \frac{x' + v_r t_{sec}'}{(1 - v_r^2/c^2)^{1/2}}$$

$$t_{sec} = (x'/c)\sinh\theta_r + t_{sec}' \cosh\theta_r = \frac{t_{sec}' + (v_r/c^2)x'}{(1 - v_r^2/c^2)^{1/2}}$$

16. Derivation of the Lorentz transformation equation. From argument (1) one obtains the condition $a + b = e + f$. Argument (2) gives the condition $b - a = e - f$. Argument (3) gives the condition $\beta_r = b/f$. From this set of three equations one obtains $f/a = 1$, $b/a = e/a = \beta_r$. Substitute these coefficients into the original equations for x and t and set up the equation for the invariance of the interval. From this equation comes the result $a = (1-\beta_r^2)^{-1/2}$. The resulting transformation equations are identical to Eqs. 16.

17. Proper distance and proper time. (a) Set the x' axis along the direction between the two events as observed in the laboratory frame. Assume that a rocket frame exists in which the two events occur at the same time. Then, from the Lorentz transformation equation,

$$\Delta t' = 0 = -\Delta x \sinh\theta_r + \Delta t \cosh\theta_r$$

from which

$$\sinh\theta_r/\cosh\theta_r = \tanh\theta_r = \beta_r = \Delta t/\Delta x < 1$$

Since the magnitude of $\Delta t/\Delta x$ is less than one, the speed β_r between frames is less than one, so the postulated rocket frame exists. From the invariance of the interval we have

$$(\Delta x)^2 - (\Delta t)^2 = (\Delta x')^2 - 0^2 = (\Delta \sigma)^2$$

so that the separation between the two events in this rocket frame is equal to the proper distance between the events.

(b) Again set the x' axis along the direction between the two events as observed in the laboratory frame; this time assume that a rocket frame exists in which the two events occur at the same place. Then

$$\Delta x' = 0 = \Delta x \cosh \theta_r - \Delta t \sinh \theta_r$$

from which

$$\tanh \theta_r = \beta_r = \Delta x / \Delta t < 1$$

so that such a rocket frame exists. Notice that $\Delta x/\Delta t$ is just the speed in the laboratory frame necessary to carry the rocket observer from one event to the other. No such interpretation is involved in part a. From the invariance of the interval

$$(\Delta t)^2 - (\Delta x)^2 = (\Delta t')^2 - 0^2 = (\Delta \tau)^2$$

so the time between these two events in the particular rocket frame is equal to the proper time between them.

18. The place where both agree. There are two ways to do this problem, one involving a small amount of talk, the other involving a large amount of mathematical manipulation! The verbal argument is as follows. The plane in which laboratory and rocket clocks agree must be perpendicular to the direction of relative motion, because it is only in such a plane that clocks are observed to be in relative synchronization by both laboratory and rocket observers (part b of Ex. 11). Now, the laboratory and rocket frames are in every way equivalent. Therefore the speed of the "plane of agreement" must be the same (with a possible difference of direction) observed in the rocket frame as observed in the laboratory frame. What intermediate speed is the same in magnitude observed from both reference frames? Not $\beta/2$. Something moving with speed $\beta/2$ as observed in the laboratory frame is not observed to have speed $-\beta/2$ as observed in the rocket frame (velocities do not add). However, something moving with velocity parameter $\theta_r/2$ in the laboratory frame will have velocity parameter $-\theta_r/2$ as observed in the rocket frame (velocity parameters do add). The velocity of the "plane of agreement" in the laboratory frame is therefore $\beta = \tanh(\theta_r/2)$, assuming such a plane exists.

The mathematical manipulation leading to the same result is the following. Set $t = t'$ in the Lorentz transformation equations 36. Eliminate x' between these equations and find an expression for the quantity x/t, the velocity of the plane of equal time. The result is (see Table 8)

$$x/t = \frac{\cosh \theta_r - 1}{\sinh \theta_r} = \frac{2 \sinh^2(\theta_r/2)}{2 \sinh(\theta_r/2) \cosh(\theta_r/2)} = \tanh(\theta_r/2)$$

19. Transformation of angles. Call $\Delta x'$ the projection of the meter stick on the x' axis as observed in the rocket frame. Call $\Delta y'$ the projection on the rocket y' axis. The tangent of the angle ϕ' is then $\tan\phi' = \Delta y'/\Delta x'$. As observed in the laboratory frame, the y projection will be the same as in the rocket frame. The x projection, however, will be Lorentz contracted (Ex. 9). We have

$$\Delta y = \Delta y' \qquad \text{with } \Delta y' = (1 \text{ meter}) \sin\phi'$$

and

$$\Delta x = \Delta x'(1 - \beta_r^2)^{1/2} \qquad \text{with } \Delta x' = (1 \text{ meter}) \cos\phi'$$

From this result the tangent of the angle in the laboratory frame can be calculated

$$\tan\phi = \Delta y/\Delta x = \tan\phi'/(1 - \beta_r^2)^{1/2}$$

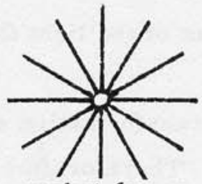
The length of the "meter" stick as observed in the laboratory frame is

$$L = \left[(\Delta x)^2 + (\Delta y)^2\right]^{1/2}$$

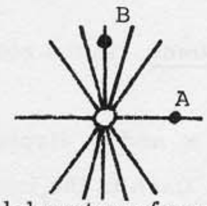
Substitute from above and obtain

$$L = \left[1 - \beta_r^2 \cos^2\phi'\right]^{1/2}$$

Treating electric field lines as meter sticks, one obtains the following configurations of electric field around a charged particle at rest in the rocket frame as observed in rocket and in laboratory frames.



rocket frame



laboratory frame

We assume that the electric force exerted on a charged test particle at rest in the laboratory frame is proportional to the density of electric field lines at the location of the test particle. It follows that test charges lying along the direction of motion of the fast particle (e.g., at point A in the figure) will experience less force than they would if the source charge were at rest; test charges off the line of motion of the test particle will experience, at the instant of closest approach (e.g., point B in the figure), a force greater than they would experience if the source particle were at rest. An excellent book by E. M. Purcell published by the McGraw-Hill Company bases the study of electricity and magnetism in large part on this and related relativistic phenomena.

20. Transformation of y velocity. From the statement of the problem we know that the separation $\Delta x' = 0$ between any two events on the world line of the particle. Therefore, from the Lorentz transformation equations

$$\Delta y = \Delta y'$$

$$\Delta x = \Delta t' \sinh\theta_r$$

$$\Delta t = \Delta t' \cosh\theta_r$$

from which the velocity components in the laboratory frame may be calculated,

$$\beta^y = \Delta y/\Delta t = \Delta y'/(\Delta t' \cosh \theta_r) = \beta^{y'}/\cosh \theta_r$$

$$\beta^x = \Delta x/\Delta t = \tanh \theta_r$$

as was to be shown.

21. Transformation of velocity directions. In the rocket frame the displacements are given by the equations

$$\Delta y' = \beta' \sin \phi' \Delta t'$$

$$\Delta x' = \beta' \cos \phi' \Delta t'$$

Find the laboratory displacements Δy and Δx using the Lorentz transformation equations 42. Then the angle made by the velocity vector in the laboratory frame is

$$\tan \phi = \Delta y/\Delta x = \frac{\beta' \sin \phi'/\cosh \theta_r}{\beta' \cos \phi' + \beta_r}$$

The angle differs from the angle found in Ex. 19 because in the present exercise one is transforming velocity, in which time enters. As $\beta_r \rightarrow 1$, the angle $\phi \rightarrow 0$ in the equation above. In contrast, the angle of the meter stick $\rightarrow 90^\circ$ as $\beta_r \rightarrow 1$ in Ex. 19.

22. The headlight effect. In the rocket frame the x-displacement of the light flash is given by the equation

$$\Delta x' = \cos \phi' \Delta t'$$

Find the laboratory x and t displacements using the Lorentz transformation equations 42. The speed β of the light flash in the laboratory frame is also unity. Therefore the cosine of the angle between the light flash path and the x axis in the laboratory frame is given by the expression

$$\Delta x/\Delta t = \cos \phi = \frac{\cos \phi' + \beta_r}{\beta_r \cos \phi' + 1}$$

Trigonometric identities show the equivalence of this expression to the result of Ex. 21 in the case $\beta' = 1$. Light going into the forward hemisphere in the rocket frame corresponds to angles less than $\phi' = 90^\circ$. The expression above yields the corresponding maximum angle in the laboratory frame

$$\cos \phi = \beta_r \qquad \text{for } \phi' = 90^\circ$$

All the light emitted in the forward hemisphere in the rest frame of the lamp is concentrated into a forward cone of this angular opening as observed in the laboratory frame and as measured from the line of motion as axis.

23. Einstein's train paradox--a worked example.

24. Einstein puzzler. Yes, he will see himself. Light has the same to-and-fro velocity in his frame as in any other inertial frame. His image in the mirror will look just the same as always for any constant speed relative to the ground.

25. The pole and barn paradox. The solution to this "paradox" is that in the (rocket) frame of the runner, the front end of the pole leaves the barn before the back end of the pole enters the barn. Therefore the runner does not observe the pole to be contained entirely in the barn at any time. The detailed sequence of events is presented in the following two spacetime diagrams. Numbers in the two diagrams are derived from the following considerations. The Lorentz contraction factor is given to be 2. Therefore (Ex. 9)

$$\cosh \theta_r = 2$$

From the identity

$$\cosh^2\theta - \sinh^2\theta = 1$$

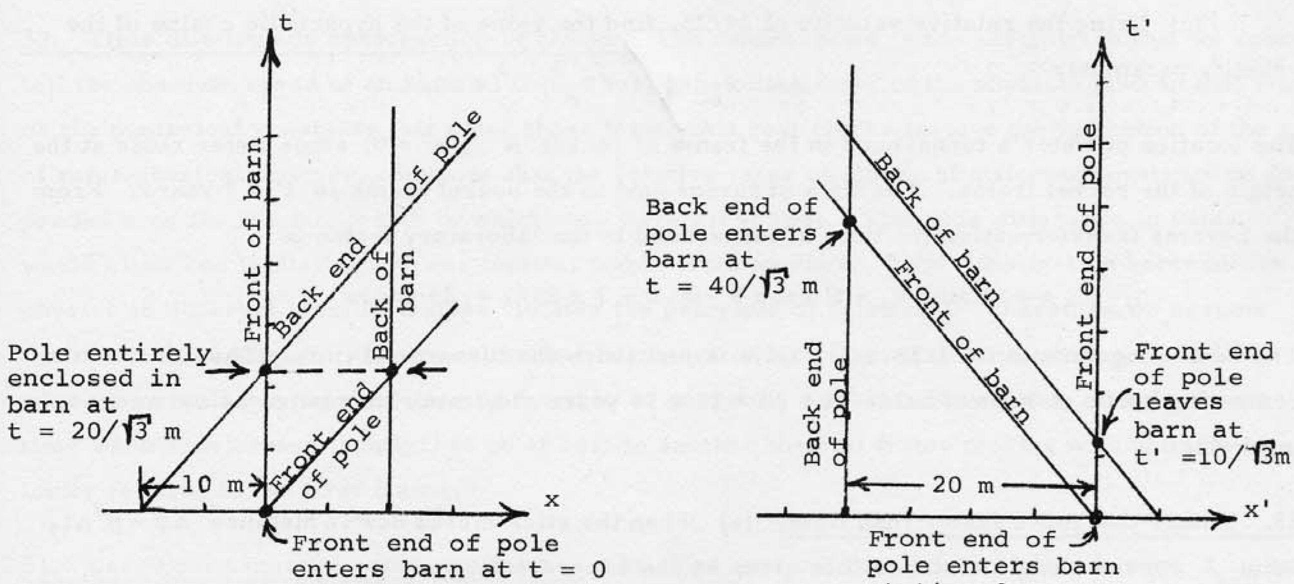
we have also

$$\sinh \theta_r = \sqrt{3}$$

Therefore the relative speed of the two frames is

$$\beta_r = \tanh \theta_r = \sqrt{3}/2$$

Use this information, plus the facts that the pole is 20 meters long as observed in the rocket frame and 10 meters long as observed in the laboratory frame to derive the numbers in the diagrams.



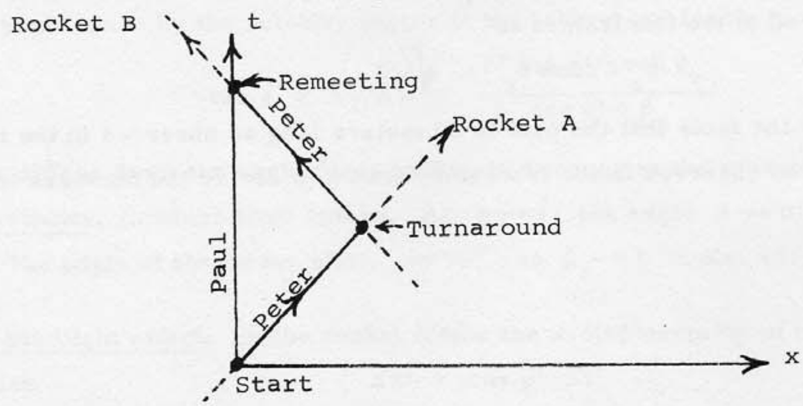
Spacetime diagram in frame of reference of barn

Spacetime diagram in frame of reference of runner

26. Space war. The difficulty lies with the concept "at the same time"--simultaneity (Ex. 11). The coincidence of points a and a' occurs at a place along the line of relative motion different from the point of firing the gun. Therefore the coincidence of points a and a' can occur at the same time as the firing of the gun in only one of the two frames. We are given that these events are simultaneous in the frame of O. Therefore figure 42 is correct by definition. However, figure 43 is incorrect: by the time points a and a' coincide in the frame of 0', the gun will already have been fired. The text phrase that introduces Fig. 43 is also incorrect. The bullet will miss the other ship as observed in both frames.

27. The clock paradox. (a) The age of traveling Peter on his return will be 21 (age at start) + 7 (spent on outbound rocket A) + 7 (spent on return rocket B) = 35 years.

(b)



(c) Using the relative velocity of 24/25, find the value of the hyperbolic cosine of the velocity parameter

$$\cosh \theta_r = 1/(1 - \beta_r^2)^{1/2} = 25/7$$

The location of Peter's turnaround in the frame of rocket A is x' = 0, since Peter rides at the origin of the rocket frame. The time of turnaround in the rocket frame is t' = 7 years. From the Lorentz transformation the time of turnaround in the laboratory frame is

$$t = x' \sinh \theta_r + t' \cosh \theta_r = 0 + 7 \times 25/7 = 25 \text{ years}$$

The remeeting time in the laboratory time is just twice the turnaround time. Therefore at the remeeting, stay-at-home Paul is 21 + 25 + 25 = 71 years old, more than twice as old as traveling Peter!

28. Things that move faster than light. (a) When the stick moves down a distance $\Delta y = \beta^y \Delta t$, point A moves along a distance Δx given by the expression

$$\Delta y/\Delta x = \tan \phi$$

from which

$$\Delta x = \Delta y/\tan \phi = (\beta^y/\tan \phi)\Delta t$$

The speed of point of intersection A is therefore

$$\beta_A = \Delta x/\Delta t = \beta^y/\tan \phi$$

For any value of β^y we can find a value of the angle ϕ near enough to zero--and yet greater than zero--so that β_A is greater than unity, that is, greater than the speed of light. This travel of the point of intersection does not carry a message, any more than a message is carried by two alarm clocks at the two locations preset to go off at times closer together than the time for light to travel from one location ot the other. In the present example, the long straight rod must be accelerated to final speed over a considerable period of time; the observer at the origin does not have the option to transmit newly-acquired information to an observer distant on the x axis by means of this intersection. An unsuccessful attempt to transmit such newly-acquired information faster than the speed of light is outlined in part (b).

(b) In this case the point of intersection can move to the right with a speed no greater than the speed of an acoustic wave in the rod, a speed very much less than the speed of light.

(c) Denote by ω the angular velocity of the searchlight in radians per second. The criterion for sweep velocity greater than c is

$$\omega r > c \quad \text{or} \quad r > c/\omega$$

The warning has not gone from one to the other,any more than a warning would go from one to the other using alarm clocks preset to go off very close together in time.

(d) Yes, writing speeds in excess of the speed of light are possible, just as sweep speeds of the searchlight in part (c) can be greater than the speed of light.

29. Synchronization by a traveling clock--a worked example.

30. Time dilation and construction of clocks. The central point is the assumption that we cannot tell the absolute speed of an inertial frame by means of the form of the physical laws in that frame or the numerical constants that enter those laws. All real clocks involve the indication of the rate of some physical process. Suppose that the relative rates of clocks of different construction depended upon the inertial frame in which they were all at rest. Then this difference in rates would allow one to distinguish one inertial frame from another. Such a distinction between the physics in different inertial frames violates the principle of relativity. Therefore we assume that it does not happen. Clocks calibrated in meters of light-travel time in one inertial reference frame will (we assume) also be found to be correctly calibrated in meters of light-travel time when accelerated (gently!) to be at rest in another inertial frame moving with uniform velocity relative to the first frame.

31. Earthbound inertial reference frames. (a) The vertical distance z fallen in time t_{sec} by a particle released from rest is given by the expression

$$z = \frac{1}{2} g t_{sec}^2$$

Here $g \approx 10$ meters per second^2 is the "gravitational acceleration" near the earth. In the present example the time is only a little more than one meter of light-travel time or $\gtrsim 3.3 \times 10^{-9}$ seconds. Therefore

$$z \approx (10/2)(3.3 \times 10^{-9})^2 \approx 5 \times 10^{-17} \text{ meters}$$

This is two orders of magnitude smaller than the size of a nucleus! Thus the region of space-time of dimensions (1 meter x 1 meter x 1 meter of space x 1 meter of time) is inertial with a sensitivity of 5×10^{-17} meters. Suppose that one measures distance of fall by interferometric techniques using visible light (for instance). Then the smallest detectable distance of fall corresponds to about one wavelength of light, say 5000 angstroms = 5×10^{-7} meters. To fall this far requires $(2 \text{ distance}/g)^{1/2} = 3 \times 10^{-4}$ seconds or 10^5 meter of light travel time. In this time a particle, moving at nearly the speed of light, could cover a spark chamber of approximate dimension $L = 10^5$ meters = 100 kilometers!

(b) In 22 meters of light travel time (73×10^{-9} seconds = 73 nanoseconds) a particle released from rest will fall a distance z approximately equal to

$$z = (10/2)(73 \times 10^{-9})^2 \approx 2.5 \times 10^{-14} \text{ meter}$$

or about three times the diameter of a nucleus. This is the sensitivity with which the earthbound frame of the Michelson-Morley experiment is inertial.

32. Size of an inertial frame. (a, 1) In Fig. 46 the right triangle with sharpest angle θ at B is similar to the right triangle with sharpest angle θ at the center of the earth. The short side of the former triangle has the length $\epsilon/2$. The short side of the latter triangle has the length (25 meters)/2. Setting up a proportion by the method of similar triangles, we have

$$\frac{(\epsilon/2)}{250 \text{ meters}} = \frac{(25 \text{ meters}/2)}{6.4 \times 10^6 \text{ meters}}$$

from which

$$\epsilon \approx 10^{-3} \text{ meters}$$

as was to be shown.

(a, 2) Consider Figure 46 with the label "25 meters" replaced by the label "Δx" and the label "r_e" replaced by the label "r". The acceleration along the line from B to the center of the earth is a^*. The component of this acceleration in the x direction (parallel to the surface of the earth) has the value $a^* \sin\theta$. The relative acceleration $(\Delta a^x)^*$ of the two particles (one dropped from B, the other dropped from A) is just the negative of twice this value

$$(\Delta a^x)^* = -2a^* \sin\theta$$

From the right triangle with smallest angle θ at the center of the earth we have

$$\sin\theta = (\Delta x/2)/r$$

so that, finally

$$(\Delta a^x)^* = -2a^*(\Delta x/2)/r = -(\Delta x/r)a^*$$

as was to be shown.

(b, 1) Use the hint.

$$(a^* \text{ at } r) = \text{const}/r^2$$

$$(a^* \text{ at } r + \Delta z) = \text{const}/(r + \Delta z)^2 = (\text{const}/r^2)(1 + \Delta z/r)^{-2} = (\text{const}/r^2)(1 - 2\Delta z/r + 3(\Delta z/r)^2 - \ldots)$$

where we have used the binomial expansion. Take advantage of the fact that Δz is very much smaller than r to neglect all terms in the binomial expansion except the first two. Then subtract the value of a^* at r from the value of a^* at $r + \Delta z$.

$$\Delta a^* \approx -2(a^*/r)\Delta z$$

The negative sign comes from the fact that the acceleration is <u>less</u> for <u>greater</u> heights. Two particles dropped from rest and separated by a vertical distance will have a relative acceleration that further separates them. This relative acceleration $(\Delta a^z)^*$ is a positive quantity of the same magnitude as Δa^*.

$$(\Delta a^z)^* \approx +2(a^*/r)\Delta z$$

as was to be shown.

(b, 2) The distance moved from rest under constant acceleration is proportional to that acceleration. Comparing Eq. 53 with Eq. 52, one sees that the relative acceleration between particles in the present case is twice the negative of that in the case treated in part a. Therefore instead of a decrease in separation of 10^{-3} meters, as in part a, one expects in the present case an <u>increase</u> in the separation of 2×10^{-3} meters. The table on page 74 requires the following revisions. In the first column, change the entry to $\epsilon \leq 2 \times 10^{-3}$ meters. Column 4 entry should read Δy and $\Delta z \leq 25$ meters. Alternatively we could leave column 1 unchanged and set $\Delta y \leq 25$ meters, $\Delta z \leq 12.5$ meters.

(c) Following the hint, we have

$$a^* \propto 1/r^2$$

$$(\Delta a^x)^* \propto \Delta x/r^3$$

$$\epsilon \propto (\Delta a^x)^*(\Delta t)^2$$

$$\epsilon \propto \frac{\Delta x(\Delta t)^2}{r^3}$$

Now ϵ is to remain the same, Δx is to increase by a factor of 8, Δt is to increase by a factor of 14, or $(\Delta t)^2$ by a factor of 200. Thus the numerator of the last fraction above is to increase by a factor of 1600, Therefore r^3 must increase by a factor of 1600 in order to keep ϵ the same value in the two cases.

$$r^3 \approx 1600\, r_e^3$$

From this result we obtain $\quad r \approx 12\, r_e$

<u>33. Michelson-Morley experiment.</u> (a) When traveling against the wind, the plane moves at a ground speed $c - v$. Therefore the time t_1 for the outward leg of the flight is $t_1 = d/(c-v)$ where d is the distance along the ground from A to B.

When traveling with the wind the plane moves at a ground speed $c + v$. Time t_2 for the return flight is $t_2 = d/(c + v)$.

The total round-trip time is $t_1 + t_2 = (2d/c)/(1 - v^2/c^2)$. But $(2d/c)$ is the time for a round trip in still air. Thus the round-trip time between A and upward point B is greater than this by a factor $1/(1 - v^2/c^2)$, as was to be shown.

A greater time is consumed going against the wind than with the wind. Therefore the time-average round-trip ground speed is less in the presence of wind than in still air. This can be seen in the limiting case in which the wind speed v is nearly equal to the airplaine air speed c. In this case the plane can return from B to A in the short time $d/(c+v) \approx d/(2v)$ but takes a very long time for the first leg of the trip upwind from A to B.

(b) In order to keep from being swept downwind, the plane must have an upwind component of the airspeed equal to the wind speed, v. The total airspeed is c. Apply the Pythagorean theorem to the velocities. Find the crosswind speed (equal to the ground speed) to be $(c^2 - v^2)^{1/2}$. The time required for a round trip of total distance 2d at this ground speed is $2d/(c^2 - v^2)^{1/2} = (2d/c)/(1 - v^2/c^2)^{1/2}$, which is a factor $1/(1 - v^2/c^2)^{1/2}$ as long as the round trip time $(2d/c)$ in still air.

(c) Let $L = 2d$ be the round-trip distance. Then the difference in time between round trips along the two perpendicular directions can be found by subtracting the "upwind-downwind" expression derived in part (a) from the "crosswind" result derived in part (b).

$$\Delta t = (L/c)(1 - v^2/c^2)^{-1} - (L/c)(1 - v^2/c^2)^{-1/2}$$

Expand the parentheses using the binomial expansion

$$\Delta t = (L/c)\left[(1 + v^2/c^2 + v^4/c^4 + \ldots.) - (1 + \frac{1}{2}v^2/c^2 + \frac{3}{8}v^4/c^4 \ldots)\right]$$

For $v/c \ll 1$ this expression is accurately approximated by the lowest power of v/c appearing in the resultant expression for Δt.

$$\Delta t \approx (L/2c)(v^2/c^2)$$

as was to be shown. The crosswind airplane will return first.

(d) Solve the above equation for v and substitute values from the statement of the problem to obtain $v = 14$ kilometers/hour. The direction of the wind lies along the line of motion of those airplanes that get back last. In which of the two directions along this line the wind blows, he cannot determine from the data given.

(e) Substituting into the equation of part (c) the value

$$L = 22 \text{ meters}$$
$$v = 30 \times 10^3 \text{ meters/second}$$
$$c = 3 \times 10^8 \text{ meters/second}$$

one obtains the value $\Delta t = (11/3) \times 10^{-16}$ second

(f) Set $\Delta t \le 10^{-2}T = 2 \times 10^{-17} \text{ sec} = (L/c)(v^2/c^2)$ (notice the cancellation of the factor 1/2 in the expression of part (c)) and substitute the given values to obtain

$$v \le 5 \times 10^3 \text{ m/sec} = (1/6)v_e$$

(g) No, the Michelson-Morley experiment by itself does not disprove the ether theory of light propagation. It could be, for instance, that the earth drags the ether along with it, so that the test apparatus is at rest in the local ether. In order to test this, one would want to try the experiment on a mountain top (it was done!) or in an earth satellite. Any well-entrenched theory requires multiple and comprehensive disproof before it is forsaken by the majority of workers in a given field of science. The Michelson-Morley experiment marked an initial blow to the ether theory from which this theory never fully recovered.

<u>34. The Kennedy-Thorndike experiment.</u> (a) In time Δt (seconds), light travels a distance $c\Delta t$ meters. In the present case this must be equal to the difference in the round-trip distances $2\Delta \ell$. Hence $\Delta t = 2\Delta \ell/c$. For $\Delta \ell = 16 \times 10^{-2}$ meters, this time difference is $\Delta t \approx 10^{-9}$ second = 1 nanosecond.

(b) $n = \Delta t/T \approx 10^{-9}/(2 \times 10^{-15}) = 5 \times 10^5$ periods. An alternative expression for n is

$$n = 2\Delta \ell/(cT)$$

(c) Suppose that n is constant (no observed shift from light toward dark in the telescope). Then c is constant provided $\Delta \ell/T$ is constant. In this sense the standard of length (the length assumed to be constant) is the dimensions of the quartz plate on which the interferometer is mounted, while the time assumed to be constant is the period of the atomic light source.

(d) Taking the differential of (54) under the assumption that $(\Delta \ell/T)$ is constant, we have

$$dc = (-2\ dn/n^2)(\Delta \ell/T)$$

or

$$dc/c = -dn/n$$

For the values given and calculated $\quad n = 5 \times 10^5$

we have $\quad dn \leq 3/1000$

$$|dc/c| \leq (3/1000)/(5 \times 10^5) = \frac{3}{5} \times 10^{-8}$$

or

$$dc \leq \frac{3}{5} \times 10^{-8} \times 3 \times 10^8 \approx 2 \text{ meters/second}$$

(as quoted in Table 4 on page 15) for the maximum change in the speed of light which could have escaped detection in this very sensitive experiment.

<u>35. The Dicke experiment.</u> (a) The copper ball falls with an acceleration g_1 and the gold ball with a slightly larger acceleration $g_2 = g_1 + \Delta g$. The difference Δg, being caused by air resistance, will be bigger near the end of the fall than at the beginning. However, we will simplify the analysis by idealizing Δg to have a certain fixed average value throughout the fall. Then the distances covered by the two balls in the same time t are

$$s_2 = (1/2)(g_1 + \Delta g)t^2$$

and

$$s_1 = (1/2)\, g_1 t^2$$

The difference is $\quad s_2 - s_1 = \Delta s = (1/2)\Delta g t^2$

Divide by the formula for the fall of the copper ball, and find

$$\Delta s/s_1 = \Delta g/g_1$$

Galileo's estimates give $s_1 = 46$ meters and $\Delta s = 7 \times 10^{-2}$ meter, or

$$\Delta g/g_1 = 7 \times 10^{-2}/46 \sim 10^{-3} \qquad \text{(Galileo)}$$

This is the maximum fractional difference in the gravitational acceleration of different objects consistent with Galileo's observations. Now suppose that this fraction has the maximum value consistent with Dicke's more recent experiment

$$\Delta g/g \leq 3 \times 10^{-11} \qquad \text{(Roll, Krotkov and Dicke)}$$

Then after the same 46-meter fall, one ball will lag behind the other by a distance

$$\Delta s = s_1(\Delta g/g_1) = 46 \times 3 \times 10^{-11} \text{ meter} = 1.5 \times 10^{-9} \text{ meter}$$

which is about ten times ~~smaller~~ LARGER than the characteristic dimension of an atom. If Δs is to be as great as 1 millimeter = 10^{-3} meters then the two balls must fall a total distance s_1 in a constant gravitational field given by

$$s_1 = \Delta s/(\Delta g/g_1) = 10^{-3}/(3 \times 10^{-11}) = 10^8/3 \text{ meters}$$

This is about one-tenth the distance from the earth to the moon (3.8×10^8 meters). Needless to say, the earth's gravitational field is not uniform to such a height.

(b) For equilibrium the net horizontal component of force must be equal to zero and the net vertical force must be equal to zero. From the figure, these conditions are satisfied when

$$T \sin \epsilon = m g_s$$
$$T \cos \epsilon = m g$$

Divide corresponding sides of the two equations to obtain

$$\tan \epsilon \approx \epsilon \approx g_s/g$$

from which

$$g_s \approx g\epsilon$$

(c) Use values inside the front cover of this book, with M equal to the mass of the sun

$$g_s = GM/R^2 = 5.94 \times 10^{-3} \text{ meter/second}^2$$

(d) Take the values from inside the front cover

$$v^2/R = 5.94 \times 10^{-3} \text{ meters/second}^2$$

In the accelerating frame of the earth this "centrifugal acceleration" away from the sun is balanced by the inward gravitational acceleration calculated in part (c). The net acceleration is zero as observed in the accelerating frame of the earth.

(e) Equation 55 comes directly from the definition of torque and inspection of Fig. 52. Use $g_s = 6 \times 10^{-3}$ meter/second^2 from section (c) to find the value of the net torque in the gravitational field of the sun

$$(\text{torque}) = (0.03 \text{ kg})(6 \times 10^{-3} \text{ meters/second}^2)(3 \times 10^{-11})(0.03 \text{ meter}) = 1.6 \times 10^{-16} \text{ kilogram meters}^2/\text{second}^2$$

A bacterium (mass 10^{-15} kilogram) at the end of a meter stick exerts a torque of approximate value

$$(10^{-5}\text{kg})(10\ \text{meters/second}^2)(1/2\ \text{meters}) \approx 5 \times 10^{-15}\ \text{kg m}^2/\text{sec}^2$$

or approximately thirty times the maximum possible torque applied to Dicke's torsion balance by the sun!

(f) Answer can be seen from Fig. 52.

(g) Equate $k\theta$ to the torque given by Eq. 55 to obtain the result given.

(h) $\theta_{tot} = 1.6 \times 10^{-8}$ radian.

36. Down with relativity! (a) See solution to Ex. 11 on time dilation.

(b) See Eq. 10 on Lorentz contraction--a worked example.

(c) One of the major results of special relativity is that the space coordinates of an event will not have the same values in a rocket frame as in a laboratory frame, and that the time lapse between two events may be different in the two inertial frames in uniform relative motion. It is no weakness of the theory that it recognizes this feature of nature. That is how the world's machinery works! If we insist on relating the observation of events to a particular frame, relativity theory can help us to find the values of coordinates in one frame, given the coordinate values in another frame. It can also relate particle velocities in one frame to velocities of the same particles as recorded in another and overlapping frame. In summary, relativity performs the following services: (1) It reveals that space coordinates individually and that time coordinates individually depend upon such an accidental circumstance as the choice of the reference frame. (2) It shows how to relate values of coordinates, velocities, accelerations, forces observed in one frame to corresponding values of these quantities as recorded in another and overlapping inertial frame. (3) It provides a "universal language"--the language of invariants--with which relations between events may be discussed independent of their space and time coordinates in any one frame. For more on this last service, see the answer to part (f) below.

(d) The equality of the speed of light in all inertial frames does indeed violate one's common sense understanding, derived from experience with the low velocities measured in everyday experience. Nevertheless, the most careful experiments have forced us to acknowledge that this apparently preposterous assertion about light is nevertheless true. In particular, the Michelson-Morley experiment (Ex. 33) and the modern revisions of this experiment have demonstrated that the speed of light is isotropic in all inertial frames. Furthermore, the Kennedy-Thorndike experiment (Ex. 34) has shown that the numerical value of this speed is the same in frames in uniform relative motion. More modern experiments now envisioned should test this conclusion with even greater sensitivity. (See text pages 14-16.)

(e) Mr. Van Dam does a service in encouraging us to sort out those predictions of relativity that have been verified directly from those predictions that have been verified indirectly or not at all. Here is a list of the status of some predictions of relativity.

Lorentz contraction (Ex. 9) The observed ionization produced by a charged particle of relativistic velocity passing through air can be satisfactorily accounted for only when one allows for the Lorentz contraction of the electric lines of force of that particle (Ex. 19). The following explanation is due to E. J. Williams (for an early and clear presentation see in particular page 331 of the article in Proceedings of the Royal Society, Series A, 130, 328, (1931). For a more analytic treatment and further references, see Proceedings of the Royal Society, Series A, 319, 163 (1933)).

> Without Lorentz contraction of the spray of electric lines of force into a thin concentrated bundle, with its plane perpendicular to the direction of motion, the charged particle could not eject electrons from atoms located far from its path, and the ionization would therefore fall far below the observed value. Consider a nitrogen atom located at the observable distance of (1/3) millimeter $\sim 3 \times 10^{-4}$ meter from the line of travel of the charged particle. In the absence of Lorentz contraction, that particle will have to move for a distance also of the rough order of magnitude of 3×10^{-4} meter for its lines of force to brush over the nitrogen atom. That will take a time (with $\beta \sim 1$) of the order of $(3 \times 10^{-4}$ meter$)/(3 \times 10^{8}$ meter/second$) \sim 10^{-12}$ second. This time of action of the electric force is far too long to affect the atom. Compare the atom with a pendulum. Move the point of support of the pendulum slowly to the right and slowly back to its original position (effect of displacement analogous to effect of electric field on atom). The pendulum will not be set into vibration by this disturbance because the effective time of action T_{force} of the force is long compared to the characteristic time of vibration T_{vib} of the pendulum. The corresponding characteristic time for the atom is 10^{-16} second. Unless the effective time of action of the electric force is short compared to this time, the atom will not be excited or ionized. The charged particle which is producing this force is already traveling at practically the speed of light so that there is no way it can shorten up its effective time of action on the nitrogen atom to less than $\sim 10^{-12}$ second--as it must, to account for the observed ionization. Here is where the Lorentz contraction comes in. It shortens the effective thickness of the bundle of lines of force, as they pass over the nitrogen atom, from $\sim 3 \times 10^{-4}$ meter to $\sim 3 \times 10^{-4} (1 - \beta^2)^{1/2}$ meter. The effective time of action of the force is shortened from $\sim 10^{-12}$ second to $\sim 10^{-12} (1 - \beta^2)^{1/2}$ second. For a charged particle with $\beta = 1 - 10^{-9}$ or $(1 - \beta^2)^{1/2} \approx (2 \times 10^{-9})^{1/2} \sim 5 \times 10^{-5}$ this time of action is $\sim 0.5 \times 10^{-16}$ second, short enough to ionize the nitrogen atom, even though it stands a million atom diameters off the line of travel of the charged particle.

Time dilation (Ex. 10). Verified with high-speed sub-atomic particles (Exs. 42, 43).

Relativity of simultaneity (Ex. 11). Verified indirectly ("Thomas precession" Ex. 103; analysis based on Ex. 52).

Clock paradox (Ex. 27). Not verified so far for clocks of everyday construction carried in space flight. Verified with significant precision for the clocks provided by the nuclei of iron atoms (Ex. 89).

The most striking and sensitive verifications of predictions unique to special relativity are to be found in the analysis of high-speed collisions, the energy balance of nuclear transformations, and the creation of pairs of particles. These reactions are discussed in the text of Chapter 2 and in the exercises of that chapter.

(f) What motorist would think of giving the latitude and longitude of each of the cities on his route? All he asks of his road map is the distance from one town to the next. Similarly in spacetime. One can dispense with coordinates, and simply list the intervals between each event and all other events. They have nothing to do with coordinates, and yet supply all the information that is really relevant.

(g) Our observations are our tie to physical reality; account for them and we have accounted for "reality" itself, insofar as that term has any scientific meaning.

37. Euclidean analogy--a worked example.

38. The Galilean transformation. Equations (57) and (58) follow from Eqs. 37 when substitutions are made of entries 4 and 5 from the right side of Table 8. In Newtonian mechanics, no distinction is made between the values of the time for the same event as measured by different observers in relative motion. In other words, Newtonian mechanics assumes that $t' = t$. Alternatively one can measure time in seconds, in which case the equation becomes $t'_{sec} = t_{sec}$. For simplicity one chooses the time $t = 0$ at the coincidence of the origins of laboratory and rocket frames. In the laboratory frame, the position of the rocket origin along the x axis as a function of time is $v_r t_{sec}$. One reasons that the rocket x-coordinate of an event is equal to the difference in coordinates between the laboratory coordinate of the event and the laboratory coordinate of the origin of the rocket frame. Hence, the expected formula is

$$x' = x - v_r t_{sec}$$

Equations 57 and 59 are nearly identical, differing only in the units in which time is measured. Notice that

$$\beta_r t = (v_r/c)t = v_r(t/c) = v_r t_{sec}$$

With this substitution, the two equations become identical. No simple substitution of units can make equations 58 and 60 identical! Rewrite Eq. 58 in terms of v_r and t_{sec}. This can be done by dividing both sides of the equation by c and recognizing that $t/c = t_{sec}$.

$$t'_{sec} = -(v_r/c)(x/c) + t_{sec} = t_{sec} - xv_r/c^2 \qquad (58')$$

The difference between Eq. (58') above and Eq. 60 in the text is the term xv_r/c^2. Under most circumstances this term can be neglected, because ordinary velocities v_r are very much less than the speed of light c. Example: The fastest man has yet traveled relative to the earth is his speed in an earth satellite, about 18,000 miles/hour = 5 miles/second = 8000 meters/second. The greatest possible distance between a rider in a satellite and an observer on earth is obtained when the man on earth is at the opposite side of the earth from the satellite. In this case the two men are separated by approximately the diameter of the earth, or approximately 13×10^6 meters. Thus the greatest value of the term xv_r/c^2 so far achievable with human observers is equal to

$$(13 \times 10^6 \text{ meters})(8 \times 10^3 \text{ meters/sec})/(3 \times 10^8 \text{ meters/sec})^2 \sim 10^{-6} \text{ second}$$

This time difference is certainly detectable by modern methods, but it is unlikely to be required in order to analyze satellite experiments, particularly since the rider in a satellite usually communicates with a ground observer on his own side of the earth!

39. Limits of accuracy of a Galilean transformation. From Table 8 one obtains the second-order approximations to the function sinh θ and cosh θ

$$\sinh\theta \approx \theta \qquad \text{(both to first order and second order)}$$
$$\cosh\theta \approx 1 + \theta^2/2$$

The second order approximation of the transformation equations 37 follows when one recalls that, even to second order, $\theta_r \approx \beta_r$

$$x' = x(1+\beta_r^2/2) - \beta_r t$$
$$t' = -\beta_r x + t(1+\beta_r^2/2) \qquad \text{(second order approximation)}$$

The coefficients of these two equations agree with the coefficients of equations 57 and 58 to better than one percent provided that $\beta_r^2/2 < 10^{-2}$

or $\beta_r^2 < 1/50$

from which, approximately $\beta_r < 1/7$

as was to be shown.

For the sports car accelerating from rest, take $a = v/t = 4$ meters/second^2. To reach $v = (1/7) \times 3 \times 10^8$ meters/second at this constant acceleration requires an approximate time $t = v/a = 10^7$ seconds, or about four months. Even at $7g \approx 70$ meters/second^2 the time required is approximately one week!

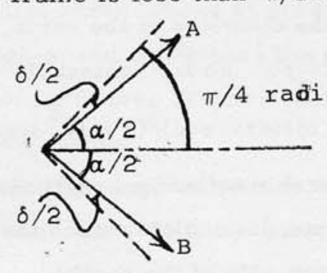
40. Collisions Newtonian and relativistic. In the rocket frame, the particles emerging from the collision travel at the velocities $\pm\beta_r$ along the y' axis. From the results of Ex. 20 (Eq. 49) we find the x and y components of their velocity in the laboratory frame.

$$\beta^y = \beta^{y'}/\cosh\theta_r = \pm\beta_r/\cosh\theta_r$$
$$\beta^x = \tanh\theta_r = \beta_r$$

The tangent of the angle $a/2$ (Fig. 53) between the x axis and either of these velocity vectors in the laboratory frame is given by the expression

$$\tan(a/2) = \beta^y/\beta^x = 1/\cosh\theta_r = (1-\beta_r^2)^{1/2}$$

We wish to discover the value of the small angle $\delta/2$ (in the figure below) by which the angle $a/2$ is less than $\pi/4$ radians. This will yield the angle δ by which the total angle a between velocity vectors in the laboratory frame is less than $\pi/2 = 90$ degrees.



Use formula 13 of Table 8

$$\tan(\delta/2) = \tan(\pi/4 - \alpha/2) = \frac{\tan(\pi/4) - \tan(\alpha/2)}{1 + \tan(\pi/4)\tan(\alpha/2)}$$

Use the expression above for tan $(\alpha/2)$ and recognize that $\tan(\pi/4) = 1$ and that for small δ, $\tan(\delta/2)$ is approximately equal to $\delta/2$. Expand the expression $(1 - \beta_r^2)^{1/2}$ using the binomial theorem and retain only the first two terms.

$$\delta/2 = \frac{1 - (1 - \beta_r^2)^{1/2}}{1 + (1 - \beta_r^2)^{1/2}} \approx \frac{1 - (1 - \beta_r^2/2)}{1 + (1 - \beta_r^2/2)} = \frac{\beta_r^2/2}{2 - \beta_r^2/2} \approx \beta_r^2/4, \qquad \delta = \beta_r^2/2$$

We are asked to find the condition on β_r such that δ is less than 10^{-2} radian. This conditions yields

$$\beta_r^2 < 1/50 \quad \text{or} \quad \beta_r < 1/7$$

For symmetrical velocities of incoming and outgoing particles in the rocket frame less than this value, the angle between the velocity vectors of outgoing particles in the laboratory frame will differ from 90 degrees by less than 10^{-2} radian. The velocity of the incident particle in the laboratory frame, in which one particle is initially at rest therefore must be less than approximately $2\beta_r < 2/7$.

41. Examples of the limits of Newtonian mechanics.

Example of motion	β	Is Newtonian analysis of this motion adequate?
Satellite circling the earth at a speed of 18,000 miles per hour.	1/37,200	Yes, because $\beta < 1/7$
Earth circling the sun at an orbital speed of 30 kilometers per second.	10^{-4}	Yes
Electron circling a proton in the orbit of smallest radius in a hydrogen atom. (Hint: The speed of the electron in the inner orbit of an atom of atomic number Z, where Z is the number of protons in the nucleus, is derived in Ex. 101 in Chap. 2 $v = (Z/137)c$ for hydrogen $Z = 1$.)	1/137	Yes
Electron in the inner orbit of the gold atom, for which $Z = 79$.	79/137	No
Electron moving with kinetic energy of 5000 electron-volts. (Hint: One electron-volt is equal to 1.6×10^{-19} joules. Try using the Newtonian expression for kinetic energy.)	4/30	Yes: on the borderline
A proton or neutron moving with kinetic energy of 10 MeV (million electron-volts) in a nucleus.	0.145 ~~10^{-2}~~	Yes; ON BORDER

42. Time dilation with μ-mesons--a worked example.

43. Time dilation with π^+-mesons. Without time dilation, half the mesons would remain at a distance of 5.4 meters from the target, under the assumptions given. According to the results of Ex. 10 (Eq. 44), $\cosh \theta_r$ is just the time dilation factor. Therefore, as observed in the laboratory frame, the π-mesons in the present experiment will appear to live 15 times as long as their "proper lifetime" as observed in the rocket frame in which they are at rest. In the laboratory these mesons are moving with nearly the speed of light. Therefore they will travel approximately 15 "characteristic distances" (see table in text), or approximately 80 meters before the beam is reduced by meson decay to half its original intensity.

44. The aberration of starlight. Let the x axis lie along the line of relative motion. In the laboratory frame, at rest with respect to the sun, the light from distant stars B and D has components of velocity $\beta^y = \pm 1$, $\beta^x = 0$. In the rocket (earth) frame, the light also moves with speed unity. However, in this rocket frame the x-component of the speed is $-\beta_r$, the relative velocity of the laboratory and rocket frame. The sine of the angle ψ is the x-component of velocity divided by the total velocity

$$\sin \psi = \beta_r / 1 = \beta_r$$

This result is consistent with the results of Ex. 22.

45. Fizeau experiment. From the law of addition of velocities Eq. 24, we have

$$\beta = (\beta' + \beta_r)(1 + \beta'\beta_r)^{-1}$$

For small β_r, expand this expression using the binomial expansion and retain terms only as high as the first power of β_r

$$(1 + \beta'\beta_r)^{-1} \approx (1 - \beta'\beta_r)$$

Substitute this into the equation above and again eliminate terms containing powers of β_r higher than the first, to obtain the answer given (Eq. 62).

46. Cerenkov radiation. Equation 63 can be read directly from Figure 62. In order to produce Cerenkov radiation in a given medium the particle must be moving at least as fast as a pulse of light in that medium. This fact is reflected in Eq. 63: the cosine of the angle ϕ can never be greater than unity. Thus in Lucite the particle must be moving with at least 2/3 the vacuum speed of light in order to produce Cerenkov radiation. On the other hand, the maximum angle ϕ in a given material will occur for the smallest value of the cosine, or the largest value of particle velocity β. Clearly β must have a value less than unity. Thus in Lucite the cosine of ϕ must have a value $(2/3)/\beta$ greater than or equal to 2/3. The maximum angle corresponding to this value of the cosine is 0.841 radian or 48.2 degrees.

47. Deflection of starlight by the sun. The time of passage of a light flash across the diameter of the sun is 1.4×10^9 meters, or 4.7 seconds. This is the "effective time of fall" of a light pulse that grazes the surface of the sun. The net velocity of fall is equal to this time multiplied by the acceleration at the surface of the sun (275 meters/second^2). This net velocity of fall is approximately 1300 meters per second, or 4.3×10^{-6} meters per meter of light-travel time. The angle of deflection for small deflections is approximately this velocity of fall divided by the total velocity of the light pulse (value unity!). This analysis thus leads to a prediction of 4.3×10^{-6} radians for the angle of deflection. Twice this value, as predicted by general relativity, agrees well with the observed values quoted at the end of the exercise.

48. Geometric interpretation. This exercise is written in such a way that each conceptual step is a small one and the reader is led through to the solution. It is probably not worthwhile to present here the logical sequence in even greater detail. In the last part, part j, it is useful to point out that the amount by which clocks in laboratory and rocket frames fail to be synchronized is governed by the value of $\sinh \theta_r$ (Eq. 46), which reverses sign when the relative velocity (and thus the relative velocity parameter) reverses sign. In contrast the amount of time dilation is governed by the value of $\cosh \theta_r$ (Eq. 44), which does not reverse sign when the relative velocity changes sign.

49. The clock paradox II--a worked example.

50. Contraction or rotation? (a) Light reaches the eye here and now from two events that took place at different distances from the eye. The events must therefore have occurred at different times. This is the central point. In particular (part a) light must leave E one meter of time earlier than light leaves G if both are to arrive at the observer simultaneously. In this time, the cube at rest in the rocket frame moves a distance x equal to β times 1 meter.

(b) The interesting thing about one-eyed visual observation of small objects under these circumstances is that all of the sightings can be interpreted as a rotation of the passing objects. Thus if the cube were tilted as in Fig. 74, one would expect to see part of the back side and a foreshortened lower edge, effects explained in relativity by the finite speed of propagation of light and the Lorentz contraction respectively. From the figure, the angle ϕ of this apparent rotation is given by the expression

$$\sin \phi = \beta$$

In the limiting case $\beta \rightarrow 0$, the angle of apparent rotation goes to zero also, and the everyday Newtonian conditions of observation obtain. In the limiting case $\beta \longrightarrow 1$, the object appears to rotate through 90 degrees, so one sees only the backside as it is seen to pass overhead!

(c) To the observer in the rocket frame: "When an object is at rest in a given frame, the method of observing it does not really matter, since time lags in the observation of different parts do not introduce distortion into the resulting picture."

To the observer using the laboratory latticework of clock: "Your clocks allow you to record the times of widely separated events and to determine cleanly whether they are coincident. This sharpness of recording does not give you license to label as "unreal" the results obtained by the observer in the rocket frame--or those obtained by the far away visual observer 0."

To the visual observer at rest in the laboratory frame: "If you understand the effects of time delay in the reception of signals from different points on the object, you understand why your visual impression of the inclination of the object does not correspond to that obtained by either of the other observers."

The word "really" does not have here a unique meaning independent of observer's frame of reference and method of measurement. All methods of measurement are "valid," but some are more useful than others in guiding intuition and predicting the results of one or another specific experiment.

51. Clock paradox III. This problem comes close to being a worked exercise! (a) If Newtonian mechanics were correct, then after 10 years of acceleration at one g, the final velocity would be

$$v = at = gt \approx (10 \text{ meters/second}^2)(10 \times 3 \times 10^7 \text{ seconds}) \approx 3 \times 10^9 \text{ meters/second}$$

or ten times the speed of light! The alternative to this physically impossible result is presented in the text of the exercise.

(b) Worked in the exercise.

(c) Equation 66 is most easily verified by taking its differential and comparing with the previous equation. Taking the differential may be simplified by expressing the hyperbolic sine and cosine in terms of exponentials (Table 8).

(d) Making the substitutions recommended in Eq. 66, we obtain

$$x = \frac{c^2}{g}\left[\cosh\left(\frac{g\tau_{sec}}{c}\right) - 1\right]$$

Substitute the approximate values $g \approx 10$ meters/second^2, ten years $\approx 3 \times 10^8$ seconds. Use an approximation from Table 8 to find

$$x \approx \frac{9 \times 10^{16}}{10}\left[\cosh\left(\frac{10 \times 3 \times 10^8}{3 \times 10^8}\right) - 1\right]$$

$$\approx 9 \times 10^{15} (e^{10}/2) \text{ meters}$$

$$\approx 10^{20} \text{ meters}$$

$$\approx 10^4 \text{ light years}$$

This is distance covered in the first or "A-jet" phase of the trip. Most remote point reached is at twice this distance, or about 20,000 light years.

52. The tilted meter stick. The answer to this exercise hinges on the relativity of simultaneity (Ex. 11). In the laboratory frame every point on the meter stick crosses the x axis simultaneously

at $t = 0$. Not so as observed from the rocket frame! At laboratory time $t = 0$, clocks on the positive x' rocket axis are reading times less than zero (part c of Ex. 11). This means that at rocket time $t' = 0$ the front end of the meter stick has already passed the x' axis. But the middle of the meter stick crosses the rocket origin at $t' = 0$. Therefore as observed in the rocket frame the meter stick is tilted upward to the right, as shown in Fig. 77. In terms of symbols, the right end of the rod as observed in the laboratory frame crosses the x axis at $t = 0$ and at the position $x = 1/2$ meter. The coordinates of this event in the rocket frame are found by using the Lorentz transformation equations

$$x' = x \cosh \theta_r = \frac{1}{2} \cosh \theta_r \text{ meter}$$

$$t' = -x \sinh \theta_r = -\frac{1}{2} \sinh \theta_r \text{ meter}$$

We want to find the position of the right end of the meter stick not at negative time $t' = -x \sinh \theta_r$, but at time $t' = 0$, which is $x \sinh \theta_r = \frac{1}{2} \sinh \theta_r$ meters later. To what position has the right end of the meter stick moved in this elapsed time? The velocity components of the end of the meter stick can be found from the results of exercise 20 (Eq. 49 with primed and unprimed velocity components interchanged and the velocity parameter replaced by its negative)

$$\beta^{y'} = \beta^{y}/\cosh \theta_r$$

$$\beta^{x'} = -\tanh \theta_r$$

Thus at $t' = 0$ the right end of the meter stick will be at the position

$$y' = \beta^{y'} t' = (\beta^{y}/\cosh \theta_r)(\frac{1}{2} \sinh \theta_r \text{ meters}) = (\beta^{y}/2) \tanh \theta_r \text{ meters}$$

$$x' = \frac{1}{2} \cosh \theta_r - \tanh \theta_r (\sinh \theta_r)/2 = \frac{1}{2} (\cosh \theta_r - \frac{\sinh^2 \theta_r}{\cosh \theta_r}) = 1/(2 \cosh \theta_r)$$

The center of the meter stick is crossing the rocket origin at $t' = 0$. Therefore the angle ϕ between the meter stick and the rocket x' axis is given by the expression

$$\tan \phi = y'/x' = \beta^{y} \sinh \theta_r$$

53. The meter-stick paradox. There will be no collision. In the rocket frame, to be sure, the meter is not Lorentz contracted. Nevertheless, in the rocket frame the rising plate is tilted, with its right hand end uppermost. In fact, Figure 77 can be thought of as is a picture of the hole in the plate! The right hand end of this hole slips neatly over the leading edge of the horizontal meter stick and the left hand end of the hole over the trailing edge of the meter stick. In this way a contracted hole held at an angle fits over a full-length meter stick.

54. The thin man and the grid. The key idea is this: There is no such thing as a "rigid" meter stick--or a "rigid" bridge. Let a long bridge be supported at both ends. Let the right hand support suddenly be removed. The right hand end starts to fall at once. Not so the middle of the

bridge. It knows nothing about the removal of the right hand support. A man standing in the middle finds his feet on iron as solid as ever. That iron starts to fall only after a certain time delay. That time delay is governed by the time required for an elastic wave to move through iron from the right hand end of the bridge to the place where the man is standing. Similarly with the meter stick. Of course there is the possibility to increase the rigidity of the meter stick by finding improved materials of construction. In this way the speed of the elastic wave can be increased, and the time reduced before the middle of the stick starts to fall. However, there is a limit to this improvement process. The speed of the elastic wave can never exceed the speed of light. The time can never be reduced below the travel time of light.

To have disposed of the misleading concept of rigidity helps to clarify another otherwise paradoxical situation. A meter stick is lying on a narrow ledge in the rocket. The ledge suddenly collapses and the meter stick falls with the acceleration of gravity. All parts of the meter stick fall with the same timing in the rocket frame. Not so in the frame of reference of the laboratory. Relative to the laboratory the rocket is shooting by to the right, parallel to the ledge, at high speed. In the laboratory frame of reference the right hand end of the meter stick starts to fall first--while the left hand side is still lying on the ledge. The meter stick appears bent--and <u>is</u> bent--as recorded in the laboratory frame of reference. Yet this bending contradicts no relativistic valid concept of "rigidity." Thus a meter stick can look straight in one frame and bent in another.

The solution of the apparent paradox is now clear: The meter stick falls through the hole. As seen in the laboratory frame this conclusion was already natural. The meter stick was Lorentz contracted to something much less than a meter, and therefore easily fell through the hole. In the rocket frame the hole was contracted to something much less than a meter, and the stick had its full length. However, we now recognize that the meter stick was not and could not be rigid. Its right hand end bent down and entered the hole, and the rest of the meter stick came following after.